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Mathematical Reviews

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HISTORY

van der Waerden, Bartel L. Eine byzantinische Sonnentafel. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1954**, 159–168 (1955).

Huber, Peter. Zu einem mathematischen Keilschrifttext (VAT 8512). Isis **46**, 104–106 (1955).

Buch, Kai Rander. When calculus of probability became science. Nordisk Mat. Tidskr. **3**, 19–26, 80 (1955). (Danish. English summary)

Archibald, R. C. The first published table of logarithms to the base ten. Math. Tables Aids Comput. **9**, 62–63 (1955).

Pelseneer, Jean. Peut-on planifier la recherche scientifique? Ce que nous dit l'histoire des mathématiques. Isis **46**, 95–98 (1955).

***Pastor, Julio Rey. Modern mathematics in Latin America.** Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Julio, 1954, pp. 9–20. Centro de Cooperación Científica de la UNESCO para América Latina, Montevideo, Uruguay, 1954. (Spanish)

Kalmár, L. L'influence de la géométrie de Bolyai-Lobatchevsky sur le développement de la méthode axiomatique. Acta Math. Acad. Sci. Hungar. **5**, supplementum, 117–126 (1954). (Russian summary)

Published originally in Hungarian [Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **3**, 235–242 (1953); MR **15**, 383].

Valentinuzzi, Maximo. Florentino Ameghino as a mathematical biologist. An. Soc. Ci. Argentina **158**, 4–34 (1954). (Spanish)

Schrek, D. J. E. David Bierens de Haan. Scripta Math. **21**, 31–41 (1955).

Aleksič [Alexits], G. Life and activity of János Bolyai. Acta Math. Acad. Sci. Hungar. **5**, supplementum, 1–20 (1954). (Russian)

Russian version of a paper originally in Hungarian [Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **3**, 131–150 (1953); MR **15**, 383].

***Carathéodory, Constantin. Gesammelte mathematische Schriften. Bd. 2.** Herausgegeben im Auftrag und mit Unterstützung der Bayerischen Akademie der Wissenschaften. C. H. Beck'sche Verlagsbuchhandlung, München, 1955. xi+457 pp. (2 plates). DM 46.00.

[For Bd. 1, see MR **16**, 434.] This volume contains 26 papers on calculus of variations, thermodynamics, geometrical optics, and mechanics. Of the papers on calculus of variations, two deal with the early history of the subject, and the remaining four (dated 1910–1933) with (i) the circle as figure of maximum content for given perimeter (with E. Study), (ii) Delaunay's problem, (iii) closed extremals and periodic variational problems, (iv) curves with bounded curvature. On thermodynamics there are two papers: (i) on the foundations (1909), (ii) determination of energy and absolute temperature from reversible processes (1925). On geometrical optics, eight papers (1926–1949): (i) the absolute optical instrument, (ii) canonical transformations, (iii) geometry of congruences of rays, (iv) parabolic reflecting telescope, (v–vi) the Schmidt telescope, (vii) aberrations of higher order, (viii) calculation of diffraction curves from the eikonal. Of these, (vi) and (viii) are printed from manuscripts, the latter with introductory remarks by M. Herzberger. On mechanics, ten papers (1923–1947): (i) on Hencky-Prandtl curves (with E. Schmidt), (ii) axioms of special relativity, (iii) the brachistochrone, (iv) sleighs (as non-holonomic systems), (v) Lagrange's exact solutions of the three-body problem, (vi) the motion of a particle under various types of constraint, (vii) on space-time (encyclopedia article), (viii–ix) the Kepler problem, (x) the two-body problem (this last from manuscript). There is a photograph of the author, and also a photograph of a page of manuscript. All the papers are in German, except three in French and one in English.

J. L. Synge (Dublin).

Gabba, Alberto. Le trasformazioni cremoniane in una lettera di Luigi Cremona a Giovanni Schiaparelli. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **18(87)**, 290–294 (1954).

Breus, K., and Položiĭ, G. Obituary: Vadim Evgen'evič D'yačenko. Ukrain. Mat. Ž. **6**, 367–370 (1954). (Russian)

A list of D'yačenko's published papers is included.

McCrea, W. H., and Lawson, Robert W. Obituary: Albert Einstein. Nature **175**, 925–927 (1955).

Mihaĭlov, G. K. Leonhard Euler. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1955**, no. 1, 3–26 (4 plates) (1955). (Russian)

Biography and discussion of Euler's work; also a large bibliography.

Levey, Martin. Solomon Gandz, 1884–1954. Isis **46**, 107–110 (1955).

Tenca, Luigi. Relazioni fra Guido Grandi e Giulio Carlo Fagnani. Atti Accad. Sci. Ist. Bologna. Cl. Sci. Fis. Rend. (11) **1**, no. 2, 77–87 (1954).

Keller, Ott-Heinrich, und Engel, Wolfgang. Heinrich Wilhelm Ewald Jung. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe **4**, 417–421 (1955).

The mathematical treatises of Omar Khayyām. Istor.-Mat. Issled. **6**, 9–112 (1953). (Russian)

Rozenfel'd, B. A., and Juškevič, A. P. Notes to the mathematical treatises of Omar Khayyām. Istor.-Mat. Issled. **6**, 113–172 (1953). (Russian)

These two papers are respectively Russian translations from the Arabic and commentaries to (1) Khayyam's treatise on the solution of the cubic by intersecting conics, (2) his work on the postulates of Euclid, and (3) a short tract applying the classical Archimedean method of determining the proportion of gold and silver in an alloy of the two.

For (1) the translator used the text established by Woepcke and published with the French translation of 1851 [L'algèbre d'Omar Alkhayyâmî, Duprat, Paris]. He remarks the translation into English by Kasir [Columbia Univ., 1931], but does not mention the more recent one of Winter and 'Arafat [J. Roy. Asiatic Soc. Bengal. Sci. **16**, 27–77 (1950); MR **13**, 809]. The four versions differ only in insignificant details.

The original of (2) is a work in three parts composed in 1077. The translator used the edition of T. Erani [Sirousse, Teheran, 1936, with Persian and Arabic introductions] based on a presumably unique Leyden manuscript. However a second copy, in the library of the Sepahsalar Mosque of Teheran, was utilized by D. E. Smith [Scripta Math. **3**, 5–10 (1935)] who noted that the first part of (2) is an attempt to prove the postulate of parallels, and that it was Khayyam and not Saccheri who initiated the study of the "birectangular quadrilateral". The second and third parts of (2) [also studied by Plooij, Thesis, Univ. of Leiden, 1950; MR **15**, 383] are a criticism of the euclidean definitions of ratio and proportion. As an alternative Khayyam defines both in terms of sequences obtained by application of the euclidean algorism. Thus (2) is a welcome addition to the scanty material in European languages on medieval Islamic work in the foundations of mathematics.

E. S. Kennedy (Beirut).

van Dantzig, D. Laplace, probabiliste et statisticien, et ses précurseurs. Arch. Internat. Hist. Sci. **8**, 27–37 (1955).

Kárteszi, F. La vie et les oeuvres de N. I. Lobatchevsky. Acta Math. Acad. Sci. Hungar. **5**, supplementum, 127–136 (1954). (Russian summary)

Published originally in Hungarian [Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **3**, 189–197 (1953); MR **15**, 384].

Gaĭduk, Yu. M. On the history of the struggle for the recognition in Russia of the geometrical ideas of Lobačevskiĭ. Ukrain. Mat. Ž. **6**, 476–478 (1954). (Russian)

***Lukomskaya, A. M. Aleksandr Mihailovič Lyapunov. Bibliografiya. [Aleksandr Mihailovič Lyapunov. A bibliography.]** Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1953. 268 pp. (4 plates). 8.20 rubles.

The bibliography is preceded by a sketch of Lyapunov's life by V. I. Smirnov (pp. 11–18), a survey of his scientific work by Smirnov (pp. 19–88), and a survey of work by Soviet mathematicians on the theory of stability by N. P. Erugin (pp. 89–96). The bibliography is divided into three parts: I) the published work of Lyapunov; II) material pertaining to his life and work; III) fundamental papers by Soviet mathematicians on the theory of stability of motion. Smirnov and V. P. Basov also took part in selecting and annotating the papers in III; this list spans the period 1929–1952 but the compilers disclaim any pretension to completeness. There is also an index to names occurring in the text.

Koyré, Alexandre. Pour une édition critique des oeuvres de Newton. Rev. Hist. Sci. Appl. **8**, 19–37 (1955).

Taton, René. L'"Essay pour les coniques" de Pascal. Rev. Hist. Sci. Appl. **8**, 1–18 (1955).

Have

***Oeuvres de Henri Poincaré.** Publiées sous les auspices de l'Académie des Sciences par la Section de Géométrie. Tome X. Publié avec la collaboration de Gérard Petiau. Gauthier-Villars, Paris, 1954. x+632 pp. (1 plate).

This volume completes the publication of Poincaré's published papers. It also completes the publication, begun in vol. IX, of his papers on mathematical physics. The papers are arranged into two groups: Oscillations hertziennes and Critiques, discussions et exposés sur les théories physiques. Each section is followed by a brief commentary by G. Petiau. For earlier volumes see MR **13**, 421, 810; **15**, 227; **16**, 435.

Saltykow, N. Henri Poincaré (1854–1912). Srpska Akad. Nauka. Zb. Rad. **43**. Mat. Inst. **4**, 1–13 (1955). (Serbo-Croatian. French summary)

Wunderlich, Herbert. Das Dresdner "Quadratum geometricum" aus dem Jahre 1569 von Christoph Schissler d.Ä., Augsburg. Wiss. Z. Tech. Hochsch. Dresden **4**, 199–227 (1955).

Drobot, S. The work of Jan Śniadecki in the mathematical and natural sciences. Wiadom. Mat. (2) **1**, 95–111 (1955). (Polish)

Tenca, Luigi. Relazioni fra Vincenzio Viviani e Michel Angelo Ricci. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **18(87)**, 212–228 (1954).

FOUNDATIONS

Have

***Dürr, Karl. Lehrbuch der Logistik.** Verlag Birkhäuser, Basel-Stuttgart, 1954. viii+181 pp. Broschiert Sw. Fr. 19.80; ganzleinen Sw. Fr. 22.90.

This book is an introductory text on logic. It is divided into three parts. Part 1 first describes the language of the propositional calculus. Here and throughout the rest of the book, the distinction between an object and its name is carefully maintained. The notation used is that of Łukasiewicz which avoids the use of parentheses. Next the propositional calculus is presented using the truth-table method. Part 2 deals with the predicate calculus and the extended predicate calculus. The monadic and polyadic predicate calculi are presented in a manner due to Quine [J. Symb. Logic **10**, 1–12 (1945); MR **7**, 45] as follows: First the monadic predicate calculus is introduced. This has long been known to be decidable, and Quine has simplified the decision procedure to a workable level. This procedure is used here. All schemata of the polyadic predicate calculus which are provable in the system of Hilbert-Ackermann can then be deduced from monadically valid schemata by substitution and the following rule of generalized modus ponens: If a conditional is valid, and its antecedent consists of zero or

more quantifiers followed by a valid schema, then its consequent is valid. (A schema is monadically valid if it is obtainable by substitution in a valid monadic schema.) Here the author elaborates this method to yield also the extended predicate calculus. Part 3 contains a section on the theory of identity, one on descriptions, and lastly a development of the virtual theory of classes and relations. In the appendix there are lists of the theorems whose detailed proofs appear in the text. It is perhaps unfortunate that there is no discussion of Gödel's completeness theorem [Monatsh. Math. Phys. **37**, 349–360 (1930)], nor of Church's theorem [J. Symb. Logic **1**, 40–41, 101–102 (1936)], since the emphasis on the decision procedure for the monadic predicate calculus raises the question whether this can be extended to the polyadic case. Nonetheless the approach is interesting and clearly presented and should prove very valuable.

I. Novak Gál (Ithaca, N. Y.).

Rasiowa, H., and Sikorski, R. On existential theorems in non-classical functional calculi. Fund. Math. **41**, 21–28 (1954).

Let S_x be the Heyting propositional calculus, S_x^* the Heyting functional calculus. The authors here prove the following theorem: If the formula $\alpha+\beta$ is provable in S_x^*, then either α or β are provable in S_x^*. If the formula $\sum_{x_p}\alpha$ is provable in S_x^*, then there is a positive integer q such that the formula $\alpha\binom{x_q}{x_p}$ is provable in S. This is a generalization of a result first stated by Gödel [Ergebn. Math. Kolloq. **4**, 39–40 (1933)]. Corresponding theorems for the positive functional calculus, the minimal functional calculus and the Lewis functional calculus are also included.

I. Novak Gál (Ithaca, N. Y.).

Greniewski, Henryk. An attempt at "rejuvenation" of the square of opposition. Studia Logica 1 (1953), 276–286 (1954). (Polish)

The traditional theory of the square of opposition can be presented as a part of the sentential calculus. The square is defined as follows:

$$\text{df.}\quad \square\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} = (p_1 \supset p_2)\cdot(p_1 \not\equiv q_2)\cdot(p_2 \not\equiv q_1).$$

The dual of this function, called the antisquare, is introduced:

$$\text{df.}\quad *\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} = (\sim p_1 \cdot p_2) + (p_1 \equiv q_2) + (p_2 \equiv q_1).$$

The classical notions of "subalternatio", "propositiones contrariae", "propositiones subcontrariae", and "contradictio" are interpreted as conditional, disjunction, alternation, and symmetric difference respectively. The classical laws can be simply stated:

$$\square\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \supset (p_1 \supset p_2)\cdot(q_1 \supset q_2)\cdot(p_1 | q_1) \cdot(p_2+q_2)\cdot(p_1 \not\equiv q_2)\cdot(p_2 \not\equiv q_1).$$

Besides the classical ones several laws can be stated, e.g.,

$$\square\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \equiv \square\begin{pmatrix} q_1 & p_1 \\ q_2 & p_2 \end{pmatrix} \equiv \square\begin{pmatrix} \sim p_2 & \sim q_2 \\ \sim p_1 & \sim q_1 \end{pmatrix} \equiv \square\begin{pmatrix} \sim q_2 & q_1 \\ p_2 & \sim p_1 \end{pmatrix} \equiv \square\begin{pmatrix} p_1 & \sim p_2 \\ \sim q_1 & q_2 \end{pmatrix},$$

$$\square\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \supset (p_1\cdot p_2) + (q_1 \cdot q_2) + (p_2 \cdot q_2).$$

The law of multiplication of squares:

$$\square\begin{pmatrix} p_1 & r_1 \\ p_2 & r_2 \end{pmatrix} \cdot \square\begin{pmatrix} q_1 & s_1 \\ q_2 & s_2 \end{pmatrix} \supset \square\begin{pmatrix} (p_1+q_1) & (r_1\cdot s_1) \\ (p_2+q_2) & (r_2 \cdot s_2) \end{pmatrix}.$$

To each of these laws the dual holds. In every true square exactly two opposite sides (i.e., the horizontals or the verticals) are order-equivalent (i.e., the first elements of the sides are equivalent and the second elements are equivalent). In two true squares the left sides are order-equivalent if and only if the right sides are order-equivalent. The same holds between the upper and the lower sides. The corresponding diagonals of two true squares are simply equivalent. Examples of tautologically true squares are given. Here are two of them:

$$\square\begin{pmatrix} \sim p & (p\cdot q) \\ (p|q) & p \end{pmatrix}, \quad \square\begin{pmatrix} (p \cdot q) & (p \not\equiv q) \\ (p \equiv q) & (p|q) \end{pmatrix}.$$

A generalization (though in a somehow different direction) of the square theory to other algebraic systems can be found in W. H. Gottschalk, The theory of quaternality, J. Symb. Logic **18**, 193–196 (1953) [MR **15**, 494].

H. Hiż.

Greniewski, H. The square of opposition—a new approach. Studia Logica 1 (1953), 287–297 (1954). (Russian. English summary)

Russian version of the paper reviewed above.

Péter, Rózsa. Ein neuer Beweis für die Tatsache, dass die Klasse der primitiv-rekursiven Funktionen umfassender als die Klasse der elementaren Funktionen ist. Z. Math. Logik Grundlagen Math. **1**, 29–36 (1955).

The earlier proof of the result by I. Berecki [C. R. 1er Congrès Math. Hongrois, 1950, Akad. Kiadó, Budapest, 1952, pp. 409–417; MR **16**, 324] produced a primitive recursive function which majorized all elementary functions. This new and shorter proof is based on a use of Cantor's diagonal procedure.

I. Novak Gál (Ithaca, N. Y.).

Péter, Rózsa. Neuer Beweis dafür, dass die Klasse der Csillag-Kalmárschen elementaren Funktionen enger ist als die Klasse der primitiv-rekursiven Funktionen. Mat. Lapok **5**, 244–252 (1954). (Hungarian. Russian and German summaries)

Hungarian version of the paper reviewed above.

Cuesta, N. Deductive structures. Rev. Mat. Hisp.-Amer. (4) **14**, 104–117 (1954). (Spanish)

This contains an abstract treatment of those general properties of deductive systems which are independent of the particular axioms or rules of procedure which they involve. A closure operation on sets of elements (propositions) is taken to be fundamental, with the interpretation that a set is closed if it contains all consequences which may be deduced from any subset of its elements by means of the axioms and rules. With this notion of closure the system, called a deductive structure, may be considered either as a complete lattice or a topological space. In terms of closure the notions of postulate, base, and irreducible base are defined. Assuming the existence of a negation operation, properties of consistent subsets, which never contain both an element and its negation, are studied, as are subsets of theorems or asserted elements, as in the truth theory of Tarski. There is also a general treatment of systems obeying the laws of contradiction and excluded middle.

O. Frink (University Park, Pa.).

Wang, Hao. Undecidable sentences generated by semantic paradoxes. J. Symb. Logic **20**, 31–43 (1955).

Suppose (i) (S) is a consistent first-order (set) theory, (ii) Peano's arithmetic Z has a model in (S), the expression A^S of (S) corresponding to A of Z so that a Gödel numbering of (S) can be introduced by means of numerals of this model, (iii) a truth definition W and a designation function D for Z can be defined in (S). While the notions W and D cannot be defined in Z, (S) has an arithmetic (Skolem) model in the notation of Z, with A^Z of Z corresponding to A of (S). An application of Gödel's substitution function leads to undecided sentences involving 'W' and 'D'. The author carries out the details under the (semantic) assumption C. 3: the model of Z in (S) [cf. (ii)] is ω-satisfiable. It seems worth observing that the unpopular syntactic treatment of arithmetic models [Hilbert and Bernays, Grundlagen der Mathematik, Bd. II, Springer, Berlin, 1939, pp. 243–253] permits the replacement of C. 3 by ω-consistency, which is known to be weaker [Henkin, J. Symb. Logic **19**, 183–196 (1954); MR **16**, 103]. For, in Theorem 1, $W(\mathfrak{n})$ cannot be decided in (S) unless (Con S) can be refuted in (S), and this is excluded by ω-consistency. Similarly, in Theorem 2, for each numberal $\mathfrak{n}$, $D(\mathfrak{q}^S)=\mathfrak{n}^S$ cannot be proved in (S); but since $D(\mathfrak{q}^S)$, i.e. $\mu_m{}^S[f(m)={}^{ZS}D^{ZS}\{h[f(\mathfrak{q}^S)]\}]$, is an arithmetic term, ω-consistency ensures that, for some $\mathfrak{n}_0$, $D(\mathfrak{q}^S)\neq\mathfrak{n}_0{}^S$ cannot be proved in (S) either.

Reviewer's note. Theorem 2 contains two non-constructive features: neither $\mathfrak{n}_0$ nor the ϵ-term $D^{ZS}\{h[f(\mathfrak{q}^S)]\}$ is computed. The latter can be computed as follows: (a) the existential symbols in the axioms of (S) are replaced by function symbols whose arithmetic models, as defined in Hilbert-Bernays, are primitive recursive; (b) in the usual set theories, by the class theorem all the sets needed can be represented by means of the constants mentioned in (a), in particular, there is a term C without variables such that $D\{h[f(\mathfrak{q}^S)]\}=C$ can be proved in (S), and C^Z is primitive recursive. Thus $\mu_m{}^S[f(m)={}^{ZS}C^{ZS}]=\mathfrak{n}_0{}^S$ is undecidable in (S), i.e. either $f(\mathfrak{n}_0{}^S)={}^{ZS}C^{ZS}$ or $(m)[f(m)\neq{}^{ZS}C^{ZS}]$ is undecidable in (S). *G. Kreisel* (Reading).

Specker, E. Die Antinomien der Mengenlehre. Dialectica **8**, 234–244 (1954).

In this lecture, naive set theory is shown to be inconsistent and some ways of avoiding the antinomies are discussed.

I. Novak Gál (Ithaca, N. Y.).

Curry, H. B. Remarks on the definition and nature of mathematics. Dialectica **8**, 228–233 (1954).

Revision of a previous discourse now regarded as inadequate. The position taken here is called empirical formalism and defended against intuitionism and logicism. The concept of a formal system is discussed; mathematics is defined as the science of formal systems and thus may be conceived as an objective science which is independent of any except the most rudimentary philosophical assumptions. Criteria of acceptability for formal systems are: 1) intuitive evidence, 2) consistency, and 3) usefulness, which is the decisive consideration. *E. W. Beth* (Amsterdam).

v. Freytag Löringhoff, Bruno. Über die Bedeutung der Mathematik für die Philosophie. Studium Gen. **6**, 600–605 (1953).

Bouligand, Georges. Quelques types de situations dans la recherche mathématique. Rend. Sem. Mat. Univ. Padova **24**, 53–69 (1955).

Ren'i, Al'fred [Rényi, Alfréd]. The ideological significance of the geometry of Bolyai-Lobačevskiĭ. Acta Math. Acad. Sci. Hungar. **5**, supplementum, 21–42 (1954). (Russian)

Russian version of a paper originally published in Hungarian [Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **3**, 253–273 (1953); MR **15**, 383].

Maravall Casesnoves, Dario. Questions concerning mathematics applied to experiment. Rev. Mat. Hisp.-Amer. (4) **14**, 269–283 (1954). (Spanish)

ALGEBRA

Narayana Moorty, T. Some summation formulae for binomial coefficients. Math. Gaz. **39**, 122–125 (1955).

The various summation formulas in question are obtained by expanding $[f(x)]^{-1}$ by long division and by partial fractions and equating coefficients of x^n. Here $f(x)$ is a polynomial; the specific choices are $1-x-tx^2$, $1-x-tx^r$, $1-(t+1)x-x^r$, $1-t+x+x^2$, $1+(1-t)x+x^2$.

J. Riordan (New York, N. Y.).

Olson, Frank R. Some determinants involving Bernoulli and Euler numbers of higher order. Pacific J. Math. **5**, 259–268 (1955).

Put

$$\left(\frac{t}{e^t-1}\right)^n=\sum_{v=0}^{\infty}\frac{t^v}{v!}B_v{}^{(n)},\quad (\sec t)^n=\sum_{v=0}^{\infty}(-1)^v\frac{t^{2v}}{(2v)!}E_{2v}^{(n)}.$$

It is proved that for $i, j=0, 1, \cdots, m$,

$$|B_i{}^{(x_j)}|=\prod_{k=0}^{m}(-\tfrac{1}{2})^k\prod_{r>s}(x_r-x_s);$$

$$|E_{2i}^{(x_j)}|=\prod_{k=0}^{m}(-\tfrac{1}{2})^k\frac{(2k)!}{k!}\prod_{r>s}(x_r-x_s);$$

in particular,

$$|B_i{}^{(a+jd)}|=\prod_{k=0}^{m}\left(-\frac{d}{2}\right)^k k!,\quad |E_{2i}^{(a+jd)}|=\prod_{k=0}^{m}\left(-\frac{d}{2}\right)^k(2k)!.$$

Like results are also proved for determinants involving the numbers $C_i{}^{(j)}$ and $D_{2i}^{(j)}$ of Nörlund [Vorlesungen über Differenzenrechnung, Springer, Berlin, 1924, Ch. 6]. Application is also made to the evaluation of the determinants $|B_i{}^{(a+jd)}(x)|$ and $|B_i{}^{(a+jd)}(h(a+jd))|$. The proofs make essential use of the fact that $B_k{}^{(n)}$ is a polynomial in n of degree k; similarly for the other numbers.

In the last part of the paper similar results are obtained for determinants involving the classic orthogonal polynomials. In particular, for the Jacobi polynomial $P_n{}^{(\alpha,\beta)}(x)$ it is proved that

$$|P_i{}^{(a+jd,b+je)}(x)|=\left\{\frac{(d+e)x+d-e}{2}\right\}^{m(m-1)/2}$$

$$(i, j=0, 1, \cdots, m-1).$$

L. Carlitz (Durham, N. C.).

Carlitz, L. A special determinant. Amer. Math. Monthly 62, 242–243 (1955).

The formula

$$\begin{vmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ 1 & 2^4 & 3^4 & \cdots & n^4 \\ 1 & 2^{2n-2} & 3^{2n-2} & \cdots & n^{2n-2} \end{vmatrix} = \frac{1!3!5!\cdots(2n-1)!}{(2n-1)(n-1)!}$$

is established and some divisibility properties are noted.

W. Ledermann (Manchester).

Amato, V. Commutabilità del prodotto di una matrice per la sua derivata. Matematiche, Catania 9, 176–179 (1954).

For a complex matrix S let $U=T^{-1}ST$ be the Jordan normal form. Let R be any matrix commuting with U. Every matrix commuting with S can be written in the form TRT^{-1}. If the elements $a_{\mu\nu}$ of S are differentiable functions of t and $S'S=SS'$, then S' (i.e. matrix of the derivatives $a'_{\mu\nu}$) must be a certain matrix TRT^{-1}. There are no indications as to the actual construction of the matrices T and R.

H. Schwerdtfeger (Melbourne).

Huff, G. B. On quasi-idempotent matrices. Amer. Math. Monthly 62, 334–339 (1955).

Let A be a square matrix over C, the field of complex numbers, and let $F(x)=\sum_k F_kx^k$ be a polynomial matrix of the same order over $C[x]$. If $A^r=F(r)$ for every positive integer r, then A is said to be quasi-idempotent, $F(x)$ is called an exponential polynomial matrix associated with A, and $F^*(X)=\sum_k F_kX^k$, X being an indeterminate matrix, is called an exponential matric polynomial. It is shown that a matric polynomial $F^*(X)$ is an exponential matric polynomial if and only if $F^*(X)F^*(Y)=F^*(X+Y)$ and also that a matrix A is quasi-idempotent if and only if $A(A-E)^k=0$ for some positive integer k. If n is a positive integer such that $A(A-E)^n\neq 0$ and $A(A-E)^{n+1}=0$, then $B_0=E-(E-A)^{n+1}$ is the constant term in the exponential matric polynomial $F^*(X)$ associated with A and

$$F^*(X)=B_0+(A-B_0)X+[(A-B_0)^2/2!]X(X-E)+\cdots +[(A-B_0)^n/n!]X(X-E)\cdots(X-(n-1)E).$$

Further the coefficient of X in $F^*(X)$ is

$$B_1=(A-B_0)-(A-B_0)^2/2+(A-B_0)^3/3+\cdots +(-1)^{n-1}(A-B_0)^n/n,$$

and

$$A=B_0+B_1+B_1^2/2!+\cdots+B_1^n/n!.$$

These formulae suggest definitions of logarithmic and exponential functions of matrices. Further results embodying such definitions are obtained.

D. E. Rutherford.

Aczél, J. Remarques algébriques sur la solution donnée par M. Fréchet à l'équation de Kolmogoroff. Publ. Math. Debrecen 4, 33–42 (1955).

The author studies the general solution of the functional equation $P(s,t)P(t,u)=P(s,u)$ for finite-dimensional matrices, with $s\leq t\leq u$. Continuing earlier work by Fréchet [Méthode des fonctions arbitraires . . . , Gauthier-Villars, Paris, 1938] who supposed that the matrices were nonsingular, the author finds the general solution in all cases. The case of functions with values in an abstract group is also treated.

J. L. Doob (Urbana, Ill.).

Tartakovskiĭ, V. A. The resultant of two characteristic equations. Uspehi Mat. Nauk (N.S.) 8, no. 6(58), 127–132 (1953). (Russian)

Let Latin letter subscripts run from 1 to m, Greek from 1 to n, and let α_r, β_ρ be the characteristic roots of matrices A, B, respectively. Define new matrices by the Kronecker direct products $A'=A\times(\delta_{\rho\sigma})$, $B'=(\delta_{rs})\times B^T$, B^T the transpose of B. The author then shows that the characteristic roots of $A'-B'$ are then $\alpha_r-\beta_\rho$ [cf. Stéphanos, J. Math. Pures Appl. (5) 6, 73–128 (1900)].

L. C. Hutchinson.

Vivier, Marcel. Sur les annulateurs des formes extérieures. C. R. Acad. Sci. Paris 238, 548–550 (1954).

Vivier, Marcel. Note sur les sommes directes de multivecteurs. C. R. Acad. Sci. Paris 240, 2285–2287 (1955).

Let $\omega_1, \cdots, \omega_n$ be disjoint simple exterior forms of degree r, spanning the space of the exterior algebra of degree nr, and set $\Delta=\omega_1+\cdots+\omega_n$. The first paper under review generalizes to even $r>2$ results concerning divisibility and divisors of zero of Δ, given by the author for $r=2$ previously [same C. R. 236, 879–881, 1725–1727 (1953); MR 15, 283]. The second paper discusses for complex field the case of any $r>2$, odd or even, showing for example that, contrary to the case of $r=2$, the decomposition $\Delta=\omega_1+\cdots+\omega_n$ is unique.

L. C. Hutchinson (Boston, Mass.).

Ślebodziński, W. Algorithm of exterior forms. Prace Mat. 1, 71–92 (1955). (Polish. Russian and English summaries)

The paper gives information on the fundamental concepts and theorems of the exterior form calculus and surveys applications in the theory of Pfaff's equations, in differential geometry, in the theory of groups and the theory of integral invariants.

Author's summary.

Spampinato, Nicolò. Le funzioni totalmente derivabili nell'algebra dei numeri n-potenziali. Atti del Quarto Congresso dell'Unione Matematica Italiana, Taormina, 1951, vol. II, pp. 220–231. Edizioni Cremonese, Roma, 1953.

✶Sawyer, W. W. Prelude to mathematics. Penguin Books, 1955. 214 pp. $.65.

This sequel to "Mathematician's delight" [Penguin Books, 1943] was worked out while the author was head of the mathematics department of University College, Gold Coast, where, although "of course, there was no mathematical tradition in the country", he found the students "keen and of first-rate ability". Defining mathematics as "the classification and study of patterns," he makes a good attempt to show the general reader how vast the subject has become. He gives a glimpse of such varied topics as Hilbert's Finite-Basis Theorem, the hypergeometric function, quaternions and Cayley numbers, the Galois group of an equation, finite geometries and their application to statistics, Riemann's finite but unbounded space, and instances of "the help that geometry gives to algebra". Though avoiding technical details, he never sacrifices accuracy. His genial personality shows itself in many places. He follows the example of Sir Harold Jeffreys in beginning each chapter with a quotation, such as Cayley's advice in 1855 about the "theory of matrices which, it seems to me, ought to come before the theory of determinants;" or, setting the stage for projective geometry, "Infinity is where things happen that don't"; or, a propos of hyperbolic geometry represented by

the interior of a sphere, "I could be bounded in a nutshell and count myself a king of infinite space". There are 85 good figures, and hardly any misprints. It may be hoped that the second edition will be supplied with an index.

H. S. M. Coxeter (Toronto, Ont.).

Abstract Algebra

***Moisil, Gr. C. Introducere in algebră. I. Inele şi ideale. Vol. I. [Introduction to algebra. I. Rings and ideals. Vol. I.]** Editura Academiei Republicii Populare Române, 1954. 255 pp. 9.10 Lei.

The present volume is the first one of several books that the author intends to publish, covering a wide range of algebraic topics. It consists of two parts. In the first part different rings (integers, residue classes, Gaussian integers, polynomial rings, etc.) are studied separately. The concepts of ring, module, ideal and field are introduced, but not that of a group and only commutative rings with unity elements are considered. The Euclidean algorithm and the unique factorization is carefully discussed. Examples of non-euclidean quadratic fields are given. A chapter is devoted to the study of congruences, Wilson's and Fermat's theorems, quadratic and cubic residues, the quadratic reciprocity law (without proof). Mention is made of Pell's equation, of the resultant of two polynomials, of Tschirnhaus transforms and of the discriminant of a polynomial. The chapters of this first part are: I) The ring of integers; II) Residue class rings; III) Gaussian integers; IV) Integers in $R(i\sqrt{3})$; V) Quadratic irrationalities; VI) to IX) Polynomials with complex, real, rational, and integral coefficients, respectively; X) Congruences of higher order; XI) Polynomials in several variables.

In the second part of the book an abstract study of rings (commutative and with unity), ideals and fields is made. The approach is axiomatic, but the author frequently refers to the concrete examples presented in the first part. The imbedding of an integral domain into its field of quotients, quotient rings, algebraic extension fields, principal ideals, conjugate ideals, principal ideal rings and the concepts of isomorphism, homomorphism and equivalence classes are discussed. The chapters are: I) Definition of rings, fields and ideals; II) The ring of quotients of a ring; III) The residue class ring; V) Adjunction of an irrational quantity; VI) Rings with factorization into primes; VII) Properties of ideals; VIII) Ideals of $I/(m)$; IX) Principal ideal rings; X) Rings with conjugate ideals; XI) Isomorphism and homomorphism; XII) Equivalence relations.

Every abstract concept introduced is generously illustrated by concrete examples and a number of problems are included in the text. The restrictions of commutativity and the requirement of a unity element for rings somewhat narrow down the scope of the book; also lead to statements like: "The ring of natural integers has no proper subrings", etc. It may also be observed, that the sets of axioms used are not always independent (for rings the commutativity is postulated and then both associative laws are postulated and numbered separately). As it stands, the book represents a very valuable contribution to the mathematical literature in Romanian. *E. Grosswald* (Philadelphia, Pa.).

Belousov, V. D. On distributive systems of operations. Mat. Sb. N.S. **36**(78), 479–500 (1955). (Russian)

Denote by $A, B, C, \cdots$ binary operations defined on an arbitrary set M, i.e., $A(a, b) \in M$ for $a, b, \in M$. The operation A is said to be right invertible if $A(a, x)=b$ has a unique solution x for all $a, b \in M$. The operation A is defined to be left distributive with respect to the operation B if $A[a, B(b, c)]=B[A(a, b), A(a, c)]$. If P is a field, the operations $Ap(a, b)\equiv(1-p)a+pb$, for all $p\neq 0 \in P$, form a system of right invertible, left distributive operations denoted by $\Pi(P)$. If G is a group, the operations $Bq(a, b)\equiv ba^{-1}qa$, for all $q \in G$, form a system of right invertible, left distributive operations denoted by $\Pi(G)$.

The author considers the structure of a system R of right invertible, left distributive operations defined on a set M. First, R forms a group under the product definition $AB(a, b)=A[a, B(a, b)]$ with unit element E, where $E(a, b)=b$ for all $a, b \in M$. If in addition, for any three elements $a\neq b\neq c$ of M, there exists in R an operation A such that $A(a, b)=c$ and an operation B such that $B(a, a)=a$ for all $a \in M$, the author shows that it is always possible to define a field structure P on M in terms of operations in R such that $R\equiv\Pi(P)$. If, however, there exists an operation B in R such that $B(a, a)\neq a$ for all $a \in M$, then it is possible to define a group structure G on M in terms of the operations in R such that $R=\Pi(G)$. The author also considers the question of uniqueness (to within an isomorphism) of P and G. *L. J. Paige* (Los Angeles, Calif.).

Schützenberger, Marcel Paul. Théorie combinatoire des relations bilinéaires classiques. Bull. Sci. Math. (2) **79**, 12–32 (1955).

Let ρ be a symmetric binary relation on a set E of "points", and define $P+Q$ for any subsets P, Q of E as $(P\cup Q)^{**}=(P^*\cap Q^*)^*$ [in the notation of the reviewer's "Lattice theory", Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, Ch. IV, §5; MR **10**, 673]. Call ρ a "classic bilinear relation" when: (i) for any distinct points $a, b \in E$ and any $c \in E$, $c^*\cap(a+b)>0$, and (ii) if $c\neq a$ is contained in $a+b$, then $a+c=b+c$ (Exchange Axiom). It is then known that the "closed" subsets $P=P^{**}$ of E form an abstract affine geometry (special "matroid" lattice). The author develops this idea further, defining hyperbolic and parabolic lines, direct sums, and quadrics. Though the conclusions are fragmentary, the discussion contains various interesting original remarks. *G. Birkhoff.*

Jakubík, Ján. On the graphical isomorphism of semi-modular lattices. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **4**, 162–177 (1954). (Slovak. Russian summary)

It is shown that if two finite lattices are graphically isomorphic (i.e., their unordered graphs are isomorphic) and one is modular then so is the other, and hence they are lattice-isomorphic. If S_1 and S_2 are graphically isomorphic semi-modular lattices, and if all sublattices of a certain type are preserved under the isomorphism, then there exist lattices A, B for which the following lattice-isomorphisms of direct products hold: $S_1\simeq A\times B$, $S_2\simeq A'\times B$, where A' is the dual of A. This is not true if one omits the requirement of preservation of certain sublattices, contrary to the author's result for modular lattices [Čehoslovack. Mat. Ž. **4**(79), 131–141 (1954); MR **16**, 440]. *P. M. Whitman.*

Szász, G. On weakly complemented lattices. Acta Sci. Math. Szeged **16**, 122–126 (1955).

A weakly complemented lattice [Areškin, Dokl. Akad. Nauk SSSR (N.S.) **90**, 485–486 (1953); MR **15**, 193] is alternatively defined as a lattice L with 0 in which for each pair $u, v \in L$ with $u<v$, there exists at least one semi-complement of u which is not a semi-complement of v;

x is called a semi-complement of u if $x \cap u = 0$. Complementation of a lattice does not imply weak complementation, nor vice versa. Relative complementation and existence of 0 imply weak complementation, which in turn implies semi-complementation, and any complemented modular lattice is weakly complemented (but not so if modularity is replaced by semi-modularity). Thus weak complementation is a generalization of relative complementation rather than of complementation. The author's remarks about the status of G. Birkhoff's Problem 73 [Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; MR **10**, 673], on correspondence between congruence relations and ideals, are perhaps a matter of interpretation. [See J. Hashimoto, Math. Japon. **2**, 149–186 (1952), especially the remarks on pp. 174–175 on the necessity of distributivity under one interpretation, and on the status under other interpretations; MR **15**, 192.] *P. M. Whitman.*

Conkling, Randall, and Ellis, David. Metric Δ-lattices. Univ. Nac. Tucumán. Rev. Ser. A. **10**, 75–82 (1954).

The authors define a Δ-lattice as a lattice with commutative multiplication, satisfying $a(b \cup c) = ab \cup ac$ and $a(b \cap c) = ab \cap ac$. If also $x \to xa$ is one-one, then the Δ-lattice is distributive. A metric Δ-lattice is one which is normed, and in which the Δ-translations are isometries. This is equivalent to having a non-negative, strictly decreasing functional $|a|$, such that $|ab| = |a| + |b|$. "Gaussian semi-groups" [or "arithmetics" in the sense of A. H. Clifford, Ann. of Math. (2) **39**, 594–610 (1938)] are characterized abstractly as a special kind of metric Δ-lattice.

G. Birkhoff (Cambridge, Mass.).

Fuchs, L. A lattice-theoretic discussion of some problems in additive ideal theory. Acta Math. Acad. Sci. Hungar. **5**, 299–313 (1954). (Russian summary)

The author considers a complete lattice L in which a binary associative but not necessarily commutative multiplication is defined possessing all essential properties of the lattice of the two-sided ideals of a general associative ring with a unit element. He generalizes the structural results concerning the additive theory of ideals in the following manner: He considers an operator ψ which is a mapping $x \to \psi(x)$ of L into itself. An element p is termed ψ-prime if $x_1 x_2 \cdots x_k \leqq p$ (k arbitrary) implies $x_i \leqq \psi(p)$ for some i. An element q is called ψ-primary if $x_1 x_2 \cdots x_k \leqq q$ (k arbitrary) implies that for each i one has either $x_i \leqq q$ or $x_j \leqq \psi(q)$ for some $j \neq i$. An element r is called right ψ-primal if $\psi(r)$ is the join of all x not right prime to r, i.e. $r : x > r$, where $r : x$ is the join of all y satisfying $yx \leqq r$. Finally, an element s is called strongly right ψ-primal or ψ^*-primal if $x \leqq \psi(s)$ is equivalent to $s : x > s$. It is shown that the following implications hold: ψ^*-primal$\Rightarrow\psi$-primal$\Rightarrow\psi$-primary$\Rightarrow\psi$-prime. In the special case where ψ is the identity mapping, all these notions coincide. Another interesting case is where $\psi(x)$ is the join of all a with the property $a^k \leqq x$ for some natural integer k. In this case ψ satisfies the following conditions: 1) $x \leqq \psi(x)$; 2) $x \leqq y$ implies $\psi(x) \leqq \psi(y)$; and 3) $\psi(x \cap y) = \psi(x) \cap \psi(y)$. In the presence of the ascending chain condition one has in addition 4) $\psi(\psi(x)) = \psi(x)$ and 5) $x \leqq \psi(y)$ implies $\psi(x) \leqq \psi(y)$. The author also shows that for this mapping the notions ψ-primary and ψ-primal coincide, and if the ascending chain condition holds then also the notions ψ-primary, ψ-primal and ψ^*-primal are identical. Also, other examples of the operator ψ are considered. For the case where $x \leqq \psi(y)$ implies $\psi(x) \leqq \psi(y)$ the author finds necessary and sufficient conditions under which the meet of a finite number of elements with a certain ψ-property (e.g. ψ-prime or ψ-primary) has again the same property. Finally, he obtains unicity theorems concerning the representation of elements of L as meets of elements having a certain ψ-property.

J. Levitzki (Jerusalem).

Szendrei, J. On the Jacobson radical of a ring. Publ. Math. Debrecen **4**, 93–97 (1955).

The author reproves a few elementary and well-known results about the Jacobson radical of a ring.

I. N. Herstein (Philadelphia, Pa.).

Szélpál, I. On the orders of elements in a module. Publ. Math. Debrecen **4**, 70 (1955).

The question considered is: given a ring R and a left-ideal L of R can we find a left-R-module G such that L is the left annihilator of some element g in G. The answer is yes, and obviously so. The author verifies this. *I. N. Herstein.*

Kertész, A. On arbitrary systems of linear equations over semi-simple rings. Bull. Acad. Polon. Sci. Cl. III. **3**, 73–74 (1955).

An announcement of the results in the paper reviewed below. *M. Henriksen* (Lafayette, Ind.).

Kertész, A. The general theory of linear equation systems over semi-simple rings. Publ. Math. Debrecen **4**, 79–86 (1955).

If R is any ring, and if A, B are arbitrary index-sets, let $R(A)$ denote the free left R-module spanned by a set $\{x_\alpha\}_{\alpha \in A}$ of indeterminates, and consider the system of linear equations

$$(1) \qquad f_\beta = b_\beta \quad (\beta \in B,\ b_\beta \in R,\ f_\beta \in R(A)).$$

Such a system is called compatible if there is an R-homomorphism of a sub-module of $R(A)$ into R sending f_β onto b_β ($\beta \in B$).

S. Gacsályi [same Publ. **2**, 292–296 (1952); MR **15**, 775] has shown that every compatible (1) has a solution in R, provided that R is a division ring. The main result of the present paper is that R is semi-simple (i.e., satisfies the descending chain condition on left ideals and contains no non-zero nilpotent left ideals) if and only if every compatible (1) has a solution in R. In this case, R admits the classical theory of linear equations in the sense of T. Szele [ibid. **2**, 297–299 (1952); MR **15**, 775]. *M. Henriksen.*

Pollák, G. Lösbarkeit eines Gleichungssystems über einem Ringe. Publ. Math. Debrecen **4**, 87–88 (1955).

Let $R[x]$ denote the free polynomial ring generated over a ring R by an arbitrary set $\{x_\lambda\}$ of indeterminates. For any index set Γ, consider the system (1) $F_\gamma = 0$ ($F_\gamma \in R[x]$, $\gamma \in \Gamma$) of algebraic equations over R. Let I denote the ideal of $R[x]$ generated by the $\{F_\gamma\}_{\gamma \in \Gamma}$. The author shows that (1) has a solution in some extension of R if and only if $I \cap R = 0$.

M. Henriksen (Lafayette, Ind.).

Guérindon, Jean. Sur les modules union ou inter-irréductibles. C. R. Acad. Sci. Paris **240**, 2042–2044 (1955).

Let A be a commutative ring with unity element and $\mathfrak{M}$ be a finitely-generated A-module. The intersection of all maximal submodules of $\mathfrak{M}$ is called the sub-radical of $\mathfrak{M}$. The module $\mathfrak{M}$ is called local if $\mathfrak{M}$ is union-irreducible in its lattice of submodules. The hypersocle of $\mathfrak{M}$, containing the usual socle, is the union of all the local submodules of $\mathfrak{M}$.

A submodule $\mathfrak{N}$ of $\mathfrak{M}$ is called superirreducible if $\mathfrak{N}$ is intersection-irreducible in the lattice of submodules of $\mathfrak{M}$. The above-defined concepts are studied for a noetherian module $\mathfrak{M}$. Sample result: $\mathfrak{M}$ is its own hypersocle if and only if the ring $A/(O:\mathfrak{M})$ is a finite direct sum of local noetherian rings. Proofs are to be given elsewhere.

R. E. Johnson (Northampton, Mass.).

Tominaga, Hisao. Some remarks on π-regular rings of bounded index. Math. J. Okayama Univ. 4, 135–141 (1955).

The author's main result is that the property of being a π-regular ring of bounded index is inherited by matrix rings. The proof uses the local finiteness of nil rings of bounded index [Levitzki, Bull. Amer. Math. Soc. 52, 1033–1035 (1946); MR 8, 435] as well as the reviewer's partial structure theory [Trans. Amer. Math. Soc. 68, 62–75 (1950); MR 11, 317]. Since it is not known whether matrix rings over nil rings are nil, it is not going to be an easy matter to delete "of bounded index" from the hypothesis and conclusion. In the final section the author shows that any ring of bounded index contains a unique largest π-regular ideal.

I. Kaplansky (Chicago, Ill.).

Steinfeld, Otto. On ideal-quotients and prime ideals. Acta Math. Acad. Sci. Hungar. 4, 289–298 (1953). (Russian summary)

An ideal p of an associative ring R is called complete prime if $ab \in p$ implies $a \in p$ or $b \in p$ where a and b are elements of R. The following theorem is proved for both prime and complete prime ideals: Let q be a non-zero ideal in the ring R, and let p be a prime (complete prime) ideal in the ring q. Then $q:p$ is a prime (complete prime) ideal in R, and $(q:p)\cap p = p$. Furthermore, if $q \neq p$, then $p:q$ is the only prime (complete prime) ideal in R whose intersection with p is p. Applications are made to almost zero-divisor free (i.e. prime) rings and to the Schreier extension theory.

F. Kiokemeister (South Hadley, Mass.).

Steinfeld, Ottó. On ideal-quotients and prime ideals. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 3, 149–153 (1954). (Hungarian)

Hungarian version of the paper reviewed above.

Kovács, István. Infinite rings without infinite proper subrings. Publ. Math. Debrecen 4, 104–107 (1955).

The author proves: Let R be an infinite ring each of whose proper subrings is finite; then R is a field of type $P_p(q^\infty)$ or is a zero-ring whose additive group is $C(p^\infty)$ where a field of type $P_p(q^\infty)$ is the union of the fields $GF(p)$, $GF(p^q)$, $\cdots$, and $C(p^\infty)$ is the Prüfer group. If only the left ideals of R are finite the R is a division ring or a zero-ring with additive group $C(p^\infty)$.

I. N. Herstein (Philadelphia, Pa.).

Behrens, Ernst-August. Ein topologischer Beitrag zur Strukturtheorie nichtassoziativer Ringe. Math. Ann. 129, 297–303 (1955).

The author extends (in a natural way) McCoy's definition of prime ideal [Amer. J. Math. 71, 823–833 (1949); MR 11, 311] to a non-associative ring $\mathfrak{o}$, and considers three structure spaces P, G, and M of $\mathfrak{o}$ consisting of all prime ideals, all F-large-modular ideals with $F(a)=(a^2-a)$ [B. Brown and N. H. McCoy, Trans. Amer. Math. Soc. 69, 302–311 (1950); MR 12, 474], and all maximal ideals, respectively, in the Stone topology. The fundamental properties established by Jacobson in the associative case [Proc. Nat. Acad. Sci. U. S. A. 31, 333–338 (1945); MR 7, 110] hold for P, G, and M. If $\mathfrak{o}$ has an identity element, then $P \supset G$ and the closure of every point in P meets G; G is dense in P if and only if $\bigcap G = \bigcap P$ and then G and P are simultaneously connected or not connected. Similar results are obtained for G and M. The author is led to observe, more generally, that a topological space E and a dense subspace U are simultaneously connected or not connected provided that the closure of each point of E meets U.

M. F. Smiley.

Fried, E. Über als echte Quotientenkörper darstellbare Körper. Acta Sci. Math. Szeged 15, 143–144 (1954).

A field K is representable as a true quotient field if K contains a domain of integrity $I \neq K$ and K is the quotient field of I. It is proved, without the use of the evaluation theory, that K is representable as a true quotient field if and only if K is not an algebraic extension field (finite or infinite) of a finite field.

F. Kiokemeister.

Hukuhara, Masuo. Théorie des endomorphismes de l'espace vectoriel. I. J. Fac. Sci. Univ. Tokyo. Sect. I. 7, 129–192 (1954).

Il s'agit en partie d'un article d'exposition; de nombreux résultats, très connus, d'algèbre linéaire, sont démontrés avec beaucoup de détails (moyennant l'hypothèse, souvent inutile, que le corps de base K est commutatif). Soient R un espace vectoriel sur K, L un endomorphisme de R, $N_n = L^n(R)$, $N^n = L^{-n}(0)$, μ (resp. ν) le plus petit entier tel que $N^\mu = N^{\mu+1}$ (resp. $N_\nu = N_{\nu+1}$) (éventuellement, $\mu = +\infty$ ou $\nu = +\infty$). Si $\mu < +\infty$ et $\nu < +\infty$, on a $\mu = \nu$, et c'est ce cas qu'étudie principalement l'A.; il existe alors des sous-espaces, non uniques, $N_k{}^j$ ($j \geqq 1$, $k \geqq 0$, $j+k \leqq \nu$), dont la somme est directe et vaut N^ν, tels que $N^j = \sum_{p \leqq j} N_q{}^p$, $N_k = N_\nu + \sum_{q \leqq k} N_q{}^p$, avec en outre

$$L(N_k{}^j) = N_{k+1}^{j-1} \quad (j>1,\ k \geqq 0,\ j+k \leqq \nu).$$

Soit R^* un espace en dualité avec R, et L^* le transposé de L, supposé exister; étude des relations qu'on peut établir entre les $N_k{}^j$ et les sous-espaces analogues relatifs à L^*. Soit $\underline{L}$ un autre endomorphisme de R, $\underline{N}^n$ et $\underline{N}_n$ les sous-espaces correspondants; l'A. dit que L est supplémentaire de L si: (i) $L\underline{L} = I$ dans N_1 et $\underline{L}L = I$ dans $\underline{N}_1$; (ii) $L(\underline{N}^n) \subset \underline{N}^{n+1}$, $\underline{L}(N^n) \subset N^{n+1}$; (iii) $L(\underline{N}_n) \subset \underline{N}_{n-1}$, $\underline{L}(N_n) \subset N_{n-1}$. L possède au moins un supplémentaire; le transposé de $\underline{L}$ est supplémentaire de L^*. Généralisation lorsque μ ou ν est infini; de nouvelles notions sont alors introduites.

J. Dixmier.

Thrall, R. M. A class of algebras without unity element. Canad. J. Math. 7, 382–390 (1955).

In a recent paper W. P. Brown [same J. 7, 188–190 (1955); MR 16, 789] studied a certain class of algebras which, as the author points out, are isomorphic to the algebra of all square matrices of degree $r+l+m$ over a field F with zeros in the first l rows and last r columns. As a generalization of these algebras, the author introduces the algebras $C = C(K, m, l, r)$ constructed in the same fashion, except that now the entries of the matrices come from a finite-dimensional division algebra K over F. Such algebras are called "submatrix algebras" and certain homomorphs of direct sums of submatrix algebras are characterized completely. Indeed, a finite-dimensional algebra A with radical N is said to be of class Q if, for some idempotent representative ϵ of the unit of A/N, (i) $\epsilon A \epsilon$ is semi-simple; (ii) $A\epsilon A = A$; (iii) $\epsilon A \epsilon + N = A$ (vector space direct sum). It is immediate that submatrix algebras are of class Q. It is then shown that for an A of class Q, $ANA = 0$, and that the class

Q is closed under the operations of taking direct sums and homomorphisms. The main theorem of the paper then asserts that every algebra of class Q is a homomorph of a direct sum of submatrix algebras, with the kernel of the homomorphism lying in the square of the radical of the direct sum.

After touching on the representation theory of algebras of class Q the author considers the problem of the uniqueness of the ϵ satisfying (i)–(iii). It is shown that if ϵ, ϵ' are two idempotents for which these conditions hold, there is an automorphism T of A carrying ϵ onto ϵ'. This automorphism is given quite explicitly in terms of two elements $\eta_0 \in \epsilon N$ and $\zeta_0 \in N\epsilon$, and indeed $\epsilon'(\epsilon+\zeta_0)(\epsilon+\eta_0)$ (Theorem 7). The group of all such T is denoted by W. Finally, the group of all automorphisms of a submatrix algebra with $K=F$ is determined: it is a product of three groups UVW with $U\cong GL(l)\times GL(r)$, $V\cong GL(m)$, and W isomorphic to a vector space of dimension m^2lr over F. All the full linear groups are over F. The problem of determining the full automorphism group of a general algebra of class Q is only begun and a connection of this problem with the theory of trilinear and quadrilinear forms is pointed out.

We should like to remark that Theorem 7 can be somewhat sharpened: It is clear that a second idempotent ϵ' satisfying (i)–(iii) leads to a new cleaving of A into the vector-space direct sum of an isomorph of A/N and N. In case A/N is separable, a theorem of Malcev [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **36**, 42–45 (1942); MR **4**, 130] asserts that for any two such cleavings there is a quasi-inner automorphism $a\to(1+n)a(1+n)^{-1}$, $n \in N$, carrying one semi-simple summand onto the other. Thus the class of idempotents satisfying (i)–(iii) is precisely the class of units of the semi-simple summands in the cleavings of A. This fact still remains true without any separability assumptions on A/N. In fact, let $A=B'+N$ be any other cleaving of A and let ϵ' be the unit of B'. Then $\epsilon'=\epsilon+n$, and it is easily verified from the fact that $\epsilon'^2=\epsilon'$ and $ANA=0$, that $n^2=n\epsilon n$. Then $(1+n\epsilon-\epsilon n)\epsilon(1+n\epsilon-\epsilon n)^{-1}=\epsilon'$, and it also follows that $(1+n\epsilon-\epsilon n)\epsilon A\epsilon(1+n\epsilon-\epsilon n)^{-1}=B'$. Hence the group W occurring in Theorem 7 is simply the group of quasi-inner automorphism by elements of N. Actually only N/N^2 is relevant in this connection since a quasi-inner automorphism by an element of N^2 is the identity. The automorphism T of Theorem 7 is simply

$$a\to(1+\zeta_0-\eta_0)a(1+\zeta_0-\eta_0)^{-1},$$

while the quasi-inner automorphism $a\to(1+n)a(1+n)^{-1}$ can be identified with a T by setting $\eta_0=-\epsilon n$, $\zeta_0=n\epsilon$. (The 1 used throughout is, of course, simply a convenient shorthand and does not mean that A has a unit.) *A. Rosenberg.*

Eilenberg, Samuel, Ikeda, Masatoshi, and Nakayama, Tadasi. On the dimension of modules and algebras. I. Nagoya Math. J. **8**, 49–57 (1955).

The main purpose of this paper is to clear up the relationships between the results and terminologies of two previous papers on the same subject; one by Eilenberg [Comment. Math. Helv. **28**, 310–319 (1954); MR **16**, 442], and the other by Ikeda, Nagao and Nakayama [Nagoya Math. J. **7**, 115–131 (1954); MR **16**, 214]. The main new result is as follows. Let A be a finite-dimensional algebra over a field K, and let I be a two-sided ideal of A which is contained in the radical of A. Then the cohomological dimension of A does not exceed the sum of the cohomological dimension of A/I and the projective dimension of A/I as a left A-module.

It was proved in the papers referred to above that, for K algebraically closed, the finiteness of the cohomological dimension of A implies that the Cartan matrix of A has determinant ±1. Here, a counterexample is given to the converse; namely an algebra whose Cartan matrix has determinant -1, but which is not of finite cohomological dimension. *G. Hochschild* (Berkeley, Calif.).

Schenkman, Eugene. On the derivation algebra and the holomorph of a nilpotent algebra. Mem. Amer. Math. Soc. no. **14**, 15–22 (1955).

Let D be the derivation algebra of a Lie algebra L. The holomorph H of L (also called elsewhere the semidirect sum of L and D) is defined to be the Lie algebra which is the vector space direct sum $H=L+D$, multiplication (commutation) being defined by

$$[l_1+D_1, l_2+D_2]=[l_1, l_2]+l_1D_2-l_2D_1+[D_1, D_2]$$

for $l_i \in L$, $D_i \in D$. It is shown that the holomorph H of L determines L up to isomorphism in case L is a free nilpotent algebra of finite class n (a free Lie algebra modulo the $(n+1)$st member of its descending central series) generated by $q\neq2$ elements. To this end relationships between D and the descending central series of L are studied.

R. D. Schafer (Storrs, Conn.).

Masuda, Katsuhiko. On a problem of Chevalley. Nagoya Math. J. **8**, 59–63 (1955).

Il est bien connu qu'une sous extension d'une extension transcendante pure $k(x_1, \cdots, x_n)$ n'est pas nécessairement une extension transcendante pure de k. Soit L le corps des invariants de l'automorphisme de $k(x_1, \cdots, x_n)$ défini par la permutation circulaire $(x_1, \cdots, x_n)$ des variables. Si k contient une racine primitive n-ième de l'unité, l'auteur montre, par extension explicite, que L est extension transcendante pure de k. Sinon une condition nécessaire et suffisante pour qu'il en soit ainsi est donnée. Cette condition est satisfaite si $n\leqq7$ et si la caractéristique p de k ne divise pas n. *P. Samuel* (Clermont-Ferrand).

Kuniyoshi, Hideo. On a problem of Chevalley. Nagoya Math. J. **8**, 65–67 (1955).

Même problème que ci dessus, dans le cas où $n=p$ (caractéristique de k). Construction explicite d'une base de transcendance de L engendrant L sur k. *P. Samuel.*

Erdös, P., Gillman, L., and Henriksen, M. An isomorphism theorem for real-closed fields. Ann. of Math. (2) **61**, 542–554 (1955).

An algebraically closed field of characteristic 0 is uniquely determined by its cardinal number. Of course this doesn't hold for a real-closed field R. An additional invariant is the order type of R, and for uncountable R this may be sufficient. The authors answer this affirmatively if the order type of R is an η_α-set: no subset of power $<\aleph_\alpha$ in R is cofinal, and if A, B are sets in R of power $<\aleph_\alpha$ with $A<B$ then an element can be interpolated. The investigation was motivated by the desire to classify the real-closed fields that show up as $C(X)/M$, where $C(X)$ is the ring of all continuous functions on a completely regular space, and M is a maximal ideal in $C(X)$. Such fields are called hyper-real if they differ from the real numbers. Sample results: assuming the continuum hypotheses, all hyper-real fields of power of the continuum are isomorphic; the hyper-real fields associated with the discrete space with the power of the continuum are not all isomorphic. The following set-theoretic lemma is

worth specially recording: for every set X of infinite power m, there exists a set of more than m subsets of X, each of power m, and such that the intersection of any two of them is of power $<m$. *I. Kaplansky* (Chicago, Ill.).

Moriya, Mikao. Zur Theorie der topologischen Körper. Math. J. Okayama Univ. 4, 115–134 (1955).

Another derivation of the known structure of locally compact division rings is given. [For references and discussion of earlier proofs, see Kaplansky, Bull. Amer. Math. Soc. 54, 809–826 (1948); MR 10, 179; and Kowalsky, Math. Nachr. 9, 261–268 (1953); MR 14, 1058.] *M. Henriksen* (Lafayette, Ind.).

Theory of Groups

Ehrenfeucht, A. On a problem of J. H. C. Whitehead concerning Abelian groups. Bull. Acad. Polon. Sci. Cl. III. 3, 127–128 (1955).

The following theorem is proved: a subgroup H of a countable free abelian group F is a direct summand of F if every homomorphism of H into the group of integers can be extended over F. *P. A. Smith* (New York, N. Y.).

Huppert, Bertram. Über Produkte von endlichen Gruppen. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe 3, 363–364 (1954).

A complete catalogue of the known cases in which, if the finite group G is the product of its subgroups H and K, a knowledge of the structure of H and K gives a knowledge of the structure of G. The following hitherto unpublished results are mentioned. First, Itô has proved that if H is dihedral and K is nilpotent then G is soluble. Secondly, the author has proved that if H is of order pq (p and q primes) and K is nilpotent then either G is soluble or it has a subgroup which has the simple group of order 168 as homomorphic image; and if H has a cyclic normal subgroup of prime index and K is abelian then G is soluble. Lastly, Itô and the author have obtained jointly results in the case when H and K are both abelian. In this case, G must have a nilpotent commutator group, and all its representations must be monomial. If H and K have mutually prime orders then G is metabelian, but it is not known whether this is true in general. *Graham Higman* (Oxford).

Huppert, Bertram. Primitive, auflösbare Permutationsgruppen. Arch. Math. 6, 303–310 (1955).

It has been conjectured that a primitive, permutation group G of degree $p+1$ (p an odd prime) is doubly transitive. Under the added hypothesis that G is solvable, the author proves the result and determines all such groups. These groups are: (1) the symmetric group S_4; (2) the group of linear transformations $y=ax+b$ with $a, b \in GF(2^r)$, $a\neq 0$, and the group of semi-linear transformations $y=ax^{2^t}+b$ with $a, b \in GF(2^r)$, $a\neq 0$, $0\leq t\leq r-1$, where $2^r-1=q$ is a prime. Several special results are obtained for primitive, solvable, permutation groups containing subgroups of particular types. *L. J. Paige* (Los Angeles, Calif.).

Pollak, G. A new proof of the simplicity of the alternating group. Acta Sci. Math. Szeged 16, 63–64 (1955). (Russian)

In this arrangement of the proof the simplicity of A_5 is established in a very direct fashion, and then induction is used to prove the simplicity of A_n. *I. Kaplansky.*

Piccard, Sophie. Les relations caractéristiques des bases du second ordre du groupe symétrique. C. R. Acad. Sci. Paris 240, 1751–1754 (1955).

Let $c_i=ba^ib^{-1}a^{-i}$ be the commutator of b and a^i. Then E. H. Moore showed that a and b generate the symmetric group S_n $(n\geq 4)$ if $a^n=b^2=(ab)^{n-1}=1$, $c_1^3=1$, $c_i^2=1$ for $i=2, \cdots, n-2$. The author shows that the relations $c_1^3=1$ and $c_i^2=1$ for $[n/2]<i\leq n-2$ may be derived from the other relations. The relation $b^2=1$ together with either $c_1=a$ or $c_1=ab$ defines S_3. Similarly five sets of 3 independent relations are given such that each set defines S_4, and 31 sets of 4 independent relations that each define S_5. *J. S. Frame* (East Lansing, Mich.).

Götlind, Erik. Some groups of order p^rq^s with Abelian subgroups of order p^r contained in the central. Ark. Mat. 3, 165–169 (1955).

Let $P=p^r$ and $Q=q^s$ be powers of distinct primes and let G_n be a group of order n. The main theorem states that if G_P is an abelian subgroup of G_{PQ} generated by two elements of different order and $p>Q$ and $p\not\equiv 1 \pmod q$, or if G_P is an abelian subgroup of G_{PQ} generated by two elements of the same order and $p>Q$ and $p^2\not\equiv 1 \pmod q$, then G_P must be contained in the central of G_{PQ}. Four lemmas are used in the proof, the first of which is a direct consequence of Sylow's theorem. *J. S. Frame* (East Lansing, Mich.).

Grün, Otto. Homomorphe Abbildungen von Gruppen auf Faktorgruppen von Untergruppen. Arch. Math. 6, 264–265 (1955).

For a finite group G, let a finite subset in one-to-one correspondence with a set of generators of G generate a subgroup U. Exchanging generators of G and U in the defining relations of G and U and then forming normal hulls $\bar{G}$ and $\bar{U}$, the author proves that $G/\bar{G}\cong U/\bar{U}$, where the isomorphism is the coset exchange induced by the exchange of generators. For a finite group G, if G can be represented as a free group modulo a word group, then each such subgroup U of G is a homomorphic image of G. Whether each homomorphic image is isomorphic to a subgroup is an open question. *F. Haimo* (St. Louis, Mo.).

Liebeck, H. A note on prime-power groups with symmetrical generating relations. Proc. Cambridge Philos. Soc. 51, 394–395 (1955).

A p-group is said to have property M if it has the maximum possible number of automorphisms; a p-group G has property E' if any automorphism of G/N can be extended to an automorphism of G (for any normal subgroup N of G). Taunt [same Proc. 51, 16–24 (1955); MR 16, 792] had previously proved that a p-group possessing property M also possesses property E' (in fact a property E stronger than E'). The present author proves the converse, namely: if G is a p-group possessing property E' then it possesses property M. *I. N. Herstein* (Philadelphia, Pa.).

Baer, Reinhold. Auflösbare Gruppen mit Maximalbedingung. Math. Ann. 129, 139–173 (1955).

A new proof of the theorem of Mal'cev [Mat. Sb. N.S. 28(70), 567–588 (1951); MR 13, 203] that for soluble groups the maximal condition for subgroups is implied by the same condition for abelian subgroups. (Soluble here means that every non-trivial homomorphic image has a non-trivial abelian normal subgroup.) First, using Dirichlet's theorem on the units of algebraic fields, one proves that an abelian group of automorphisms of a finitely generated abelian

group is finitely generated. It follows that if G satisfies the maximal condition for abelian subgroups so does G/N for any abelian normal subgroup N. Mal'cev's theorem follows if it is assumed that the soluble group in question has a term of its derived series equal to 1.

The more difficult part of the proof is to show that if a soluble group satisfies the maximal condition for abelian subgroups then some term of its derived series must be 1. This can be reduced to the statement that if a soluble group G satisfies the maximal condition for subgroups so does any soluble group Φ of automorphisms of G, which in turn, by an induction on derived length, follows from the special case when G is abelian. This again reduces to the case when G is torsion free, and Φ induces an irreducible group on the tensor product of G with the rationals. Since we already have the theorem if Φ is abelian, it is, finally, enough to show that in this case Φ has an abelian subgroup of finite index.

Accordingly, the central section of the paper deals with criteria for the existence, in any group, of soluble or abelian subgroups of finite index. For instance, a group has a soluble subgroup of finite index if and only if every infinite homomorphic image of it has an infinite normal subgroup with a finite derived group. The result needed for the application is that a soluble group G has an abelian subgroup of finite index if each normal subgroup of G induces a finite automorphism group in each of its abelian normal subgroups. There are more specialised criteria, and a related criterion for finiteness.

Reviewer's remarks. (i) Many of the author's theorems on automorphism groups of torsion-free abelian groups of finite rank become more familiar, and their proofs much easier to understand, if it is borne in mind that such groups are essentially rational matrix groups. For instance, it follows immediately from a result of Burnside [Proc. London Math. Soc. (2) 3, 435–440 (1905)] that a torsion group of this kind is finite, soluble or not. (ii) The author asserts that it is an open question whether a soluble normal subgroup N of a group G necessarily contains an abelian normal subgroup of G. It is easy to see that this is the same as asking whether N necessarily has an abelian characteristic subgroup. A counter-example, indeed an example of a soluble non-abelian characteristically simple group was given by McLain [Proc. Cambridge Philos. Soc. 50, 641–642 (1954); MR 16, 217]. McLain did not assert that his group was soluble, but he exhibited a set of abelian normal subgroups which generate it, which is more than enough for solubility in the present sense. *Graham Higman* (Oxford).

Baer, Reinhold. Burnsidesche Eigenschaften. Arch. Math. 6, 165–169 (1955).

A class $\mathfrak{C}$ of groups is called Burnsidean if every finitely generated torsion group in $\mathfrak{C}$ is finite. This is equivalent to the double requirement that every finitely generated torsion group in $\mathfrak{C}$ has only a finite number of distinct finite factor groups, and every finitely generated torsion group in $\mathfrak{C}$ that has a composition series is finite. Denote by $\mathfrak{C}'$ the class of all groups G such that every non-trivial factor group of G has a non-trivial accessible (nachinvariant) subgroup in $\mathfrak{C}$. Then if $\mathfrak{C}$ is Burnsidean so is $\mathfrak{C}'$. *Graham Higman.*

Haimo, Franklin. Power-type endomorphisms of some class 2 groups. Pacific J. Math. 5, 201–213 (1955).

A study of endomorphisms of groups of class 2 which induce endomorphisms of the form $x\to x^n$ in the factor derived group. These include $x\to(x, u)$ for any u in the group (with $n=0$) and $x\to x^n$ itself if the derived group has an exponent dividing $\frac{1}{2}n(n-1)$. If the factor group by the centre has exponent N then the endomorphisms for which N divides n form a ring. More generally, these endomorphisms, together with any power-type endomorphism whose kernel contains the derived group, generate a ring of power-type endomorphisms. If the factor derived group has exponent m, the number of endomorphisms $x\to x^j$ into the centre divides m. *Graham Higman* (Oxford).

Shenitzer, Abe. Decomposition of a group with a single defining relation into a free product. Proc. Amer. Math. Soc. 6, 273–279 (1955).

By a T-transformation on the generators $a_1, \cdots, a_n$ of the free group $F=F(a_1, \cdots, a_n)$ is meant a mapping of the form: $Ta_k=a_k$ for some fixed k, $1\leqq k\leqq n$; $Ta_i=a_i$ or $a_ia_k^\epsilon$ or $a_k^{-\epsilon}a_i$ or $a_k^{-\epsilon}a_ia_k^\epsilon$, $i\neq k$, $1\leqq i\leqq n$, where ϵ denotes either 1 or -1. A word $W(a_1, \cdots, a_n)$ is said to be minimal if the length of the word $W(Ta_1, \cdots, Ta_n)$ is not less than that of W for every T-transformation on $a_1, \cdots, a_n$. The main result of the paper is contained in the following criterion: Let the group G with generators $a_1, \cdots, a_n$ be defined by the single relation $R(a_1, \cdots, a_n)=1$, where all a_i are involved in R. Then G can be represented as a free product of an infinite cyclic group and a non-trivial group if and only if any minimal form of R contains at most $n-1$ distinct a_i's. The proof makes use of a fundamental theorem of J. H. C. Whitehead [Ann. of Math. (2) 37, 782–800 (1936), Theorem 3] and an important corollary of Grushko's theorem [A. G. Kuroš, Teoriya grupp, Gostehizdat, Moscow-Leningrad, 1944; MR 9, 267]. It is clear that if R itself is minimal then G is free indecomposable. Afterwards the author gives some simple tests for the minimality of a word R. *A. Kertész* (Debrecen).

Weir, A. J. Sylow p-subgroups of the general linear group over finite fields of characteristic p. Proc. Amer. Math. Soc. 6, 454–464 (1955).

Let K be a finite field $GF(q)$ with $q=p^k$ elements, and $GL_n(K)$ the general linear group over K. The author considers the group G of all triangular $n\times n$ matrices of the form $1+\sum_{i<j} a_{ij}e_{ij}$, where $a_{ij}\in K$ and e_{ij} is the matrix with the 1 of K in the (i, j) position and 0 elsewhere. For $i<j$ the group P_{ij} of all matrices of the form $1+ae_{ij}$ with $a\in K$ is isomorphic to the additive group of K. Subgroups of G generated by a selection of these P_{ij} are called partition subgroups and are used by the author to derive various results concerning the structure of the group G. Thus, e.g., he shows that the upper and lower central series of G coincide. Further results are obtained by using the so called diagonal automorphisms of G (i.e. automorphisms of the form $A\to W^{-1}AW$, where $A\in G$ and W is a diagonal matrix of $GL_n(K)$) and the "symmetric" automorphism $1+\sum a_{ij}e_{ij}\to 1+\sum b_{ij}e_{ij}$, where $b_{ij}=a_{n+1-j}$. The author shows that the normal partition subgroups of G may be characterized as those subgroups which are left invariant under the inner and the diagonal automorphisms, while the characteristic subgroups of G are precisely those which in addition are left invariant under the symmetric automorphism. This latter result is based upon a careful investigation of the maximal abelian normal subgroups of G. Finally, the author considers the structure of the automorphism group A of G by defining certain significant subgroups of A for which it is shown that they generate the whole of A. These latter results constitute a generalization of those which were

obtained by P. P. Pavlov [Izv. Akad. Nauk SSSR. Ser. Mat. 16, 437–458 (1952); MR 14, 533] for the special case where K is the prime field of characteristic p.

J. Levitzki (Jerusalem).

Higman, Graham. A remark on finitely generated nilpotent groups. Proc. Amer. Math. Soc. 6, 284–285 (1955).

Let G be a finitely generated nilpotent group, and let G^n be the subgroup of G generated by the nth powers of the elements of G. (The term "nilpotent" is used in the sense that the lower central series of G terminates in the identity after a finite number of steps.) It is proved that the intersection of the groups G^p is finite for any infinite set of primes p. By this theorem one obtains a short proof of a theorem of Baer [Math. Z. 59, 299–338 (1953); MR 15, 598] according to which there is an integer n such that the intersection of all characteristic subgroups of G whose indices are prime powers p^a with $a \leqq n$ is the identity.

A. Kertész (Debrecen).

Čarin, V. S. On groups of automorphisms of nilpotent groups. Ukrain. Mat. Ž. 6, 295–304 (1954). (Russian)

A multiplicative group of algebraic numbers, each a root of a polynomial over the rational field of degree not exceeding a fixed number m, is a direct product of a finite group and a free abelian group of finite or countable rank. A solvable group of non-singular matrices over the rational field contains a normal subgroup in which all the matrices are reducible to triangular form with unity on the main diagonal and such that the factor group is a finite extension of a free abelian group of finite or countable rank. Let Γ be a group of certain automorphisms of a nilpotent torsion-free group G of finite rank. If Γ is solvable, then G has a central chain of subgroups relative to a suitable normal subgroup Δ of Γ such that Γ/Δ is a finite extension of a free abelian group of finite or countable rank. If Γ is locally nilpotent, then Γ is nilpotent, it has a normal subgroup Λ of finite rank such that Γ/Λ is a free abelian group of finite or countable rank, and its maximal periodic subgroup is finite. A group having a normal series each factor of which is a non-cyclic subgroup of the additive group of rational numbers is a finite extension of a nilpotent group.

R. A. Good.

Murnaghan, Francis D. On the characters of the symmetric group. Proc. Nat. Acad. Sci. U. S. A. 41, 396–398 (1955).

If (μ) is a partition of p, the author seeks a formula for the characteristic of the class $(1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}\cdots)$ of the symmetric group of degree n corresponding to the partition $(n-p, (\mu))$. He gives a method for obtaining such a formula for any specific partition (μ), but not a general comprehensive formula. However, his methods do lead directly to such a general formula and the statement of such a formula is perhaps the most concise way of expressing the results.

Let $[\mu]$ denote the degree of the representation of the symmetric group of degree a_1 corresponding to the partition $(a_1-p_1(\mu))$. Let $\delta_j[\mu]$ denote the sum of the terms obtained by decreasing each part of the partition (μ) by j. If the parts are no longer in descending order, the usual convention is adopted for restoring the descending order.

The required characteristic is then

$$\sum \delta_2{}^b\delta_3{}^c\delta_4{}^d\cdots[\mu]\binom{a_2}{b}\binom{a_3}{c}\binom{a_4}{d}\cdots,$$

summed for all integral values of $b, c, d, \cdots$ which give non-zero terms, the expressions $\binom{a_2}{b}$, etc. denoting binomial coefficients.

D. E. Littlewood (Bangor).

Bruhat, François. Sur certaines représentations unitaires des groupes de Lie semi-simples. C. R. Acad. Sci. Paris 240, 2196–2198 (1955).

In several recent notes [see especially same C. R. 238, 38–40, 550–553 (1954); MR 15, 504, 505] the author has established the irreducibility of certain unitary representations of semi-simple Lie groups. The author now extends and modifies his methods to include the representations recently considered by Harish-Chandra [Proc. Nat. Acad. Sci. U. S. A. 40, 1076–1077, 1078–1080 (1954); MR 16, 334]. A proof is outlined to show that almost all these representations are irreducible.

F. I. Mautner (Princeton, N. J.).

Borel, A., and Chevalley, C. The Betti numbers of the exceptional groups. Mem. Amer. Math. Soc. no. 14, 1–9 (1955).

The Hirsch formula allows one to write down the Poincaré polynomial of G/U once the primitive exponents of the Poincaré polynomials of G and U are known. Here G denotes a compact connected Lie group, and U a connected compact subgroup of the same rank as G. Using this device along with divisibility properties of the primitive exponents, the authors compute the Poincaré polynomials of the exceptional Lie groups G_2, F_4, E_6, E_7. In order to compute the Betti numbers of E_8 this method is combined with some information about the invariants of the Weyl group, on which Chevalley's original method was solely based [Proc. Internat. Congress Math., Cambridge, Mass., 1950, vol. 2, Amer. Math. Soc., Providence, R. I., 1952, pp. 21–24; MR 13, 432].

W. T. van Est (Utrecht).

Gurevič, G. B. On some properties of algebraic linear Lie groups. Dokl. Akad. Nauk SSSR (N.S.) 94, 177–178 (1954). (Russian)

Using definitions and theorems from previous papers [Mal'cev, Izv. Akad. Nauk SSSR. Ser. Mat. 9, 329–356 (1945); Gurevič, ibid. 13, 403–416 (1949); MR 9, 173; 11, 156] the following theorems are proven for an algebraic linear Lie group $\mathfrak{G}$ and its corresponding Lie algebra $\mathfrak{A}$ with Θ_i the weights of $\mathfrak{G}$, Λ_i those of $\mathfrak{A}$.

1) $$\Theta_1{}^{\alpha_1}\Theta_2{}^{\alpha_2}\cdots\Theta_n{}^{\alpha_n}=\Theta_1{}^{\beta_1}\Theta_2{}^{\beta_2}\cdots\Theta_n{}^{\beta_n}$$

with α_i, β_i non-negative integers, with any other relations reducing to this form. 2) $\gamma_1\Lambda_1+\cdots+\gamma_n\Lambda_n=0$, γ_i integers. 3) The F-system of $\mathfrak{A}$ is always a null algebra. The last nicely generalizes a number of earlier special theorems of the author.

L. C. Hutchinson (Boston, Mass.).

Browne, Marjorie Lee. A note on the classical groups. Amer. Math. Monthly 62, 424–427 (1955).

Proof of the well known elementary theorem that the following pairs of groups have the same homotopy type: $U(n)$ and $GL(n, C)$; $O(n)$ and $GL(n, R)$; $SO(n)$ and $SL(n, R)$, and that $GL(n, C)$, $SL(n, C)$, $SU(n)$, $U(n)$ have same mth homotopy groups, $m>1$.

W. T. van Est.

Poncet, Jean. Sur les groupes localement compacts. C. R. Acad. Sci. Paris 240, 2476 (1955).

Rectification d'une Note antérieure [mêmes C. R. 238, 192–194 (1954); MR 15, 399].

Berikašvili, N. A. On direct and inverse spectra of topological groups. Soobšč. Akad. Nauk Gruzin. SSR **15**, 257–264 (1954). (Russian)

Topologies for the limit of a direct system of compact abelian groups have been defined by Čogošvili [Mat. Sb. N.S. **28**(70), 89–118 (1951); MR **12**, 846], S. Kaplan [Duke Math. J. **15**, 649–658 (1948); MR **10**, 233] and Vilenkin [Izv. Akad. Nauk SSSR Ser. Mat. **15**, 503–532 (1951); MR **13**, 435]. In this paper the author shows how these definitions may be derived from a common scheme.

W. T. van Est (Utrecht).

Goto, Morikuni. On the group of automorphisms of a locally compact connected group. Mem. Amer. Math. Soc. no. **14**, 23–29 (1955).

In this note the following "tower theorem" is proved: Let G be a connected locally compact group with totally disconnected centre. Let

$$A(G)=A^1(G), \cdots, A^{k+1}(G)=A(A^k(G)), \cdots$$

be the series of iterated maximal connected automorphism groups of G. ($A(G)$ is assigned the compact open topology.) Then for a certain index k, $A^{k+1}(G)\cong A^k(G)\cong$ inner automorphism group of $A^k(G)$.

The theorem for the case that G is a Lie group is due to Chevalley. The present theorem is reduced to the Lie group case by using the fact that G is the product of a continuously isomorphic image L^* of a Lie group L and a compact group K, with $[L^*, K]=(1)$ and $L\cap K$ totally disconnected. Because of the totally disconnected centre of G, L^* and K are invariant under automorphisms of G, and $A(G)$ is the direct product of $A(K)$ and a closed subgroup of $A(L)$. This leads quickly to the tower theorem. A byproduct of the analysis is that $A(G)$ is locally compact (G arbitrary locally compact and connected) if it carries L^* into itself.

W. T. van Est (Utrecht).

Takenouchi, Osamu. Sur une classe de fonctions continues de type positif sur un groupe localement compact. Math. J. Okayama Univ. **4**, 143–173 (1955).

It was first shown by V. Bargmann [Ann. of Math. (2) **48**, 568–640 (1947); MR **9**, 133] that not all irreducible unitary representations of a locally compact group G need occur in its regular representation. In the present paper this problem is studied from the point of view of positive definite functions. For this purpose the author introduces the following property R: the constant 1 on G can be approximated uniformly on every compact set by functions of the form $\int_G x(gh)\bar{x}(h)dh$ where $x \in L_2(G)$ or continuous and of compact support. The main result of the paper (Theorem 2) states: Let G be a locally compact group whose quotient by its connected component of the identity is compact. Then G has property R if and only if G is of type (C) in the sense of Iwasawa [ibid. **50**, 507–558 (1949); MR **10**, 679].

F. I. Mautner (Princeton, N. J.).

Davis, Harry F. A note on Haar measure. Proc. Amer. Math. Soc. **6**, 318–321 (1955).

The author proves the following theorem. Given any locally compact topological group G, there exists a compact subgroup K, a subspace (not necessarily a subgroup) E which is homeomorphic to an n-dimensional Euclidean space, and a discrete subset D, such that the mapping $(a, b, c)\rightarrow abc$ is a homeomorphism of the Cartesian product $D\times E\times K$ with G, and carries the product measure $m_D\times m_E\times m_K$ into m_G, where m_D is the discrete measure on D, m_E is Lebesgue measure on E relative to suitable coordinates, m_K is Haar measure on K, and m_G is left Haar measure on G.

L. H. Loomis (Cambridge, Mass.).

Følner, Erling. Note on a generalization of a theorem of Bogolioùboff. Math. Scand. **2**, 224–226 (1954).

The author improves the results of an earlier paper [Math. Scand. **2**, 5–18 (1954); MR **16**, 220] by replacing the hypothesis that the abelian group G is covered by an finite number of translates of the subset E by the weaker hypothesis that $\bar{m}(E)>0$, where the upper mean measure is the greatest lower bound of the set of least upper bounds of the various finite convex combinations of translates of the characteristic function of E.

L. H. Loomis.

Šulka, Robert. Topological groupoids. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **5**, 10–21 (1955). (Slovak. Russian summary)

A groupoid is a non-void set in which a quite arbitrary multiplication, ab, is defined for all pairs of elements a, b. A topological groupoid is a groupoid that is a T_1-space in which multiplication is continuous in both variables. The main results of this note deal with continuous homomorphic mappings of a topological groupoid G onto another, and their representation in the usual way by dissections of G.

E. Hewitt (Princeton, N. J.).

Mal'cev, A. I. Analytic loops. Mat. Sb. N.S. **36**(78), 569–576 (1955). (Russian)

This paper inaugurates the study of Lie loops: topological loops with a neighborhood of the identity homeomorphic to Euclidean space in such a way as to make the loop operations analytic. Appropriate modifications are made to define a local Lie loop. The author rapidly specializes to alternative loops, defined by the condition that any two elements generate a group. Attached to an alternative Lie loop is a binary Lie (BL) algebra: an anticommutative algebra where every two elements generate a Lie subalgebra. The usual relations hold between the loop and the algebra. Reference is made to Pontrjagin's "Topological groups" [Princeton, 1939; MR **1**, 44] with brief indications when a proof needs to be modified. Since a commutative BL-algebra has all its products 0, one derives the following corollary: a commutative alternative Lie loop is associative.

Call an anticommutative algebra Moufang-Lie (ML) if the identity $xy\cdot xz=(xy\cdot z)x+(yz\cdot x)x+(zx\cdot x)y$ is satisfied. The algebra attached to a Moufang Lie loop is ML. An intermediate concept is that of an SL-algebra, defined by postulating the Jacobi identity on xy, x and z. Any ML-algebra is SL; and for characteristic prime to 6 any SL-algebra is BL.

I. Kaplansky (Chicago, Ill.).

NUMBER THEORY

✱Sierpiński, Wacław. Arytmetyka teoretyczna. [Theoretical arithmetic.] With the cooperation of Jerzy Łoś. Państwowe Wydawnictwo Naukowe, Warszawa, 1955. 258 pp. zł. 30.

This is an excellent introductory text covering elementary number theory and an axiomatic foundation of the number system. The chapters are as follows: I. Theory of nonnegative integers (based on the axioms of Peano); II. Theory of integers and rational numbers; III. Properties of integers (divisibility, prime numbers, Euler's φ function etc.); IV. Congruences, their properties and applications; V. Real numbers (Cantor's approach); VI. Complex numbers and quaternions.

In chapter II the Erdös version of Chebyshev's proof of Bertrand's postulate is reproduced in all detail.

An American reader may be somewhat surprised at the selection of topics (usually covered here in different courses) but it seems to correspond to a course in Polish universities. There is a nice supply of problems. *M. Kac.*

Sierpiński, W. Prime numbers. Wiadom. Mat. (2) **1**, 47–64 (1955). (Polish)

Elementary expository paper.

✱Sznirelman, Lew. [Šnirel'man, L. G.] Liczby pierwsze. [Prime numbers.] Panstwowe Wydawnictwo Naukowe, Warszawa, 1954. 100 pp. zł. 3.60.

Translation by A. Götz of Šnirel'man's Prostye čisla [Gostehizdat, Moscow, 1940; MR **1**, 292].

Laborde, Pedro. A note on the even perfect numbers. Amer. Math. Monthly **62**, 348–349 (1955).

The author shows that the even perfect numbers are characterized by the property that

$$n=2^{H(n)-1}(2^{H(n)}-1),$$

where $H(n)$ is the harmonic mean of the divisors of n.

O. Ore (New Haven, Conn.).

Larsson, David F. Un théorème fondamental concernant le nombre N écrit dans la base B. Mathesis **64**, 20–22 (1955).

Let $s(n)$ be the sum of the digits of a positive integer n to base r, and let c denote the constant value assumed by the series $s(n)$, $s(s(n))$, $s(s(s(n)))$, $\cdots$ after a finite number of terms. Thus $1\leqq c\leqq r-1$, and the author proves that $n\equiv c \pmod{(r-1)}$. (The result is an immediate consequence of the well-known property $n\equiv s(n) \pmod{(r-1)}$.)

I. Niven (Berkeley, Calif.).

Bojanić, Ranko. L'évaluation asymptotique de la somme de diviseurs des certains nombres. Bull. Soc. Math. Phys. Macédoine **5** (1954), 5–15 (1955). (Serbo-Croatian. French summary)

Generalizing a problem of Erdös [Amer. Math. Monthly **61**, 350 (1954)] the author proves that if N is the set of all integers n whose least prime factor exceeds $c \log n$ then, as $n\rightarrow\infty$ over members of N, $\sigma(n)\sim n$. Examples of infinite subsets of N are (a) the Fermat numbers [Erdös' example], (b) the Mersenne numbers, (c) the Euclidean numbers $1+p_1p_2\cdots p_k$, and the numbers $2^{pq}+1$.

D. H. Lehmer (Berkeley, Calif.).

Schinzel, A. On the equation $x_1x_2\cdots x_n=t^k$. Bull. Acad. Polon. Sci. Cl. III. **3**, 17–19 (1955).

The totality of solutions in natural numbers is given in parametric form for the equation listed in the title. The case $k=2$ had been done by E. T. Bell [Ann. of Math. (2) **48**, 43–50 (1947); MR **8**, 442]. *I. Niven.*

Erdös, P. On amicable numbers. Publ. Math. Debrecen **4**, 108–111 (1955).

It is proved that the set of amicable numbers has density zero. (The numbers a and b are amicable in case the sum-of-divisors function σ satisfies $\sigma(a)=\sigma(b)=a+b$.) This improves an earlier estimate of H. J. Kanold [Math. Z. **61**, 180–185 (1954); MR **16**, 337]. *I. Niven.*

Lehmer, D. H. The distribution of totatives. Canad. J. Math. **7**, 347–357 (1955).

Totatives of n are numbers $<n$ and relatively prime to n. If k is an integer $<n$, then by

$$\phi(k, q, n) \quad (q=0, 1, \cdots, -1+n/k)$$

the author denotes the number of totatives in the interval $(nq/k, n(q+1)/k)$. If n and k are such that $\phi(k, q, n)$ does not depend on q, the totatives are said to be uniformly distributed with respect to k. Uniform distribution is proved in the following cases: (i) k^2 divides n; (ii) n is divisible by a prime of the form $kx+1$. Explicit formulas are given for $\phi(k, q, n)$ if $k=3, 4, 6$, and for all n; also for general k if n is a product of distinct primes of the form $kx-1$. Finally, the author shows that

$$|\phi(k, q_1, n)-\phi(k, q_2, n)|\leqq\theta(n)$$

and

$$|k\phi(k, q, n)-\phi(n)|\leqq(k-1)\theta(n),$$

where $\theta(n)=2^{\alpha(n)}$, and $\alpha(n)$ is the number of distinct prime divisors of n. *N. G. de Bruijn* (Amsterdam).

Gorskaya, Z. D. On an arithmetic property of a harmonic sum. Ukrain. Mat. Ž. **6**, 375–384 (1954). (Russian)

It is shown that the sum of the reciprocals of two or more integers in arithmetic progression is never an integer, except in the trivial case in which the terms annihilate each other in pairs and the sum is zero. Use is made of the Bertrand-Čebyšev theorem. *W. J. LeVeque* (Ann Arbor, Mich.).

Lorent, H. Sur l'équation indéterminée $x^3+l=y^2$. I, II. Bull. Soc. Roy. Sci. Liège **24**, 72–76, 192–197 (1955).

Mordell, L. J. Integer solutions of cubic equations in three variables. Rend. Mat. e Appl. (5) **14**, 431–438 (1955).

This is an expository paper reviewing the work of the author and others on cubic Diophantine equations in three variables with integral coefficients. *I. Niven.*

✱Katz, Stanley. On the representation of powerfree integers by systems of polynomials. Abridgment of a dissertation, New York University, 1951. 3 pp.

The following result is stated. Let $f_i(x)$ be polynomials with integral coefficients such that

$$f_i(x)=c_i\prod_{j=1}^{b_i}(g_{ij}(x))^{d_{ij}} \quad (1\leqq i\leqq a, b_i\geqq 0),$$

where the c_i are integers $\neq 0$ and each $g_{ij}(x)$ is an irreducible polynomial with integral coefficients and of content 1. Let $r_i > d_{ij} \geqq 1$; $1 \leqq i \leqq a$, $1 \leqq j \leqq b_i$, and let $\phi(n)$ be any complex-valued periodic function with an integral period. Put

$$V(z) = \sum_{1 \leqq n \leqq z} \phi(n) \prod_{i=1}^{a} \mu_{r_i}(f_i(n)),$$

where $\mu_r(n) = 1$ if N is r-free, $=0$ otherwise. Then there exists a constant λ independent of z such that

$$(*) \qquad V(z) - z\lambda = O(z) \quad (z \to \infty),$$

provided that a certain constant $\leqq 1$. Moreover, if $\phi(n) \geqq 0$ and if, for some integer n, $\phi(n) \prod_1^a \mu_{r_i}(f_i(n)) > 0$, then $\lambda > 0$. A more precise result than (*) is stated but this requires additional notation. *L. Carlitz* (Durham, N. C.).

Krajňáková, Dorota. Remark on the theory of power residues (mod p^a). Mat.-Fyz. Časopis. Slovensk. Akad. Vied 4, 212–217 (1954). (Slovak. Russian summary)

The author notes that the total number of kth power residues (mod p^a), where p is an odd prime, can be obtained from the formula

$$(1) \qquad \phi(p^a)/(k, \phi(p^a))$$

for the number of kth power residues prime to p by simply summing (1) for

$$\alpha = \alpha, \alpha - k, \alpha - 2k, \cdots.$$

The result is

$$\phi(p^{a-sk})\sigma_k(p^s)/(k, p-1),$$

where s is the greatest integer in $(\alpha-1)/k$. The density of kth power residues may thus be discussed.

D. H. Lehmer (Berkeley, Calif.).

Moisil, Gr. C. Sur un théorème de Zolotarev. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 6, 797–800 (1954). (Romanian. Russian and French summaries)

In a paper by Zolotareff [J. Math. Pures Appl. (3) 6, 51–84, 129–166 (1880)] one finds a proof of the following theorem: If the two complex numbers A and B belong to relatively prime residue classes (mod p), then there exist complex numbers M and N, such that $AM - BN \equiv 1$ (mod p). The author observes that essentially the same reasoning leads to the proof of the following more general theorem: Let Θ be a finite, commutative ring such that every element is either a unit or a divisor of zero (this last condition is clearly redundant); then, if a and b are elements of Θ, the following two properties are equivalent: (I) There exist elements h, k in Θ, such that $ha + kb = 1$; and (II) $az = bz = 0$ imply $z = 0$. (The single bibliographical reference in the paper contains at least four printing errors.)

E. Grosswald (Philadelphia, Pa.).

Carlitz, L. Congruences for generalized Bell and Stirling numbers. Duke Math. J. 22, 193–205 (1955).

Let $f(x) = \sum_1^\infty a_m x^m/m!$, where the a_m are rational integers, satisfy $f'(x) = 1 + \sum_1^\infty c_m f^m(x)$. The author [same J. 19, 329–337 (1952); MR 13, 913] has proved that the a_m satisfy congruences of Kummer's type. Now suppose that there is given a set of series $f_0(x), f_1(x), \cdots, f_s(x)$, each of the same general nature as $f(x)$. The author defines $F_k(x)$ by means of $F_0(x) = f_0(x)$, $F_k(x) = F_{k-1}(f_k(x))$ $(k = 1, 2, \cdots)$. Generalizing the numbers of Stirling and Bell, he puts

$$F_{s-1}(uf_s(x)) = \sum_{r=1}^\infty \frac{x^r}{r!} \sum_{c=1}^r u^c \sigma(c, r, s),$$

and $F_s(x) = \sum_{r=1}^\infty \beta(r, s) x^r/r!$. He shows that the numbers $\beta(r, s)$, $\sigma(c, r, s)$ satisfy congruences that include the results of Becker and Riordan [Amer. J. Math. 70, 385–394 (1948); MR 9, 568] for the generalized Bell numbers. The precise theorems of the author are somewhat elaborate to state, but they imply a number of interesting periodicity properties of the numbers $\beta(r, s)$, $\sigma(c, r, s)$ with respect to a prime power modulus. *A. L. Whiteman* (Los Angeles, Calif.).

Carlitz, L., and Olson, F. R. Maillet's determinant. Proc. Amer. Math. Soc. 6, 265–269 (1955).

Let p $(\geqq 3)$ be a fixed prime. If $(r, p) = 1$, define r' by means of $rr' \equiv 1$ (mod p). The least positive residue of r is denoted by $R(r)$. Maillet's determinant is

$$D_p = \det(R(rs')) \quad (r, s = 1, \cdots, \tfrac{1}{2}(p-1)).$$

The authors show that Malo's conjecture, viz. that $D_p = (-p)^{\frac{1}{2}(p-3)}$, is false for $p = 23$. They prove that, in fact,

$$D_p = \pm p^{\frac{1}{2}(p-3)} h,$$

where h is the first class number of the cyclotomic field $k(e^{2\pi i/p})$. This answers in the affirmative Maillet's original question whether $D_p \neq 0$. The formula for D_p was independently obtained by S. Chowla and A. Weil, who however did not publish their result.

The paper contains other interesting formulae, including one involving the Dedekind sums

$$s(r, p) = \sum_{k=1}^{p-1} \frac{k}{p} \left(\frac{rk}{p} - \left[\frac{rk}{p} \right] - \frac{1}{2} \right).$$

It is proved that

$$\det(s(rt', p)) = 2^{\frac{1}{2}(p-5)} p^{-2} h^2 \quad (r, t = 1, \cdots, \tfrac{1}{2}(p-1)).$$

W. Ledermann (Manchester).

Carlitz, L. A special determinant. Proc. Amer. Math. Soc. 6, 270–272 (1955).

The properties of Maillet's determinant [see the preceding review] suggest investigating the determinant

$$\Delta_k = \det(R((r-s)^k)) \quad (r, s = 0, \cdots, p-1; 1 \leqq k \leqq p-1)$$

Among other things it is proved that when $1 \leqq k < p-1$, the exact power of p that divides Δ_k is p^{p-k}. Also $\Delta_{p-1} = p-1$.

W. Ledermann (Manchester).

Briggs, W. E., and Chowla, S. The power series coefficients of $\zeta(s)$. Amer. Math. Monthly 62, 323–325 (1955).

Let $\zeta(s) = (s-1)^{-1} + \sum_{n=0}^\infty A_n (s-1)^n$. It is shown in two ways that $A_n = (-1)^n \gamma_n/n!$ where

$$\sum_{n=1}^{x} \frac{\log^k n}{n} = \frac{\log^{k+1} x}{k+1} + \gamma_k + o(1).$$

H. S. Zuckerman (Seattle, Wash.).

Turán, P. On Lindelöf's conjecture. Acta Math. Acad. Sci. Hungar. 5, 145–163 (1954). (Russian summary)

$N(\alpha, T)$ denotes, as usual, the number of zeros $\sigma + it$ of the Riemann zeta function in the rectangle $\alpha \leqq \sigma \leqq 1$, $0 < t \leqq T$. Various estimates of $N(\alpha, T)$ are known, for which we refer to Titchmarsh [The theory of the Riemann zeta-function, 2nd ed., Oxford, 1951, p. 196–211; MR 13, 741]. We quote A. E. Ingham's result [Quart. J. Math. Oxford Ser. 8, 255–266 (1937)] that $N(\alpha, T) < CT^{(2+\epsilon)(1-\alpha)} \log^5 T$ $(\frac{1}{2} \leqq \alpha \leqq 1, T > 2)$, assuming the Lindelöf hypothesis

$$\zeta(\tfrac{1}{2} + it) = O(t^\epsilon).$$

The present author gives a method by which results on $N(\alpha, T)$ can be obtained from information on the order of $\zeta(s)$ on vertical lines to the right of the critical line, without assuming anything about $\zeta(\frac{1}{2}+it)$. His theorem is as follows. Let $\frac{1}{2}<\vartheta<1$. Then there is a positive number $\eta_0(\vartheta)$ such that for $0<\eta<\eta_0(\vartheta)$ the following theorem holds: If

$$(1)\qquad |\zeta(\sigma+it)|\leqq c_1(\eta)\exp(\eta^{1000}\log t)\quad(\sigma\geqq\vartheta,\ t\geqq 1),$$

then we have

$$(2)\qquad N(\sigma, T)<c_2(\eta)T^{(3+8\eta)(1-\sigma)}\log^5 T\quad(\sigma\geqq\vartheta+4\eta,\ T>c_3(\eta)).$$

In the proof the author applies his power-sum method [Eine neue Methode in der Analysis und deren Anwendungen, Akad. Kiadó, Budapest, 1953; MR **15**, 688] to the sum $g_k(s)=\sum_{n\geq\xi}\Lambda(n)n^{-s}\log^k(n/\xi)$ with values of s far in the half-plane $\sigma>1$; k large, $\xi=e^{k+1}$. This is similar to a previous application [Acta Math. Acad. Sci. Hungar. **2**, 39–73 (1951); MR **13**, 742; also Th. 38 in his book just cited], which dealt with the case that σ is close to 1, and which is an auxiliary result in the present paper.

The reviewer remarks that $\log^5 T$ in (2) is an error for $\log^6 T$; this factor comes in from the case "σ close to 1". On the other hand, the $\log T$ factors can easily be disposed of entirely, by remarking that $N(\sigma, T)=O(1)$ if

$$\sigma>1-(\log T)^{-7/8}$$

[see Titchmarsh, loc. cit., p. 114]. A somewhat abbreviated exposition of these results also appears in Hungarian [Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **4**, 357–368 (1954); MR **16**, 449]. *N. G. de Bruijn* (Amsterdam).

Gundlach, Karl-Bernhard. Über die Darstellung der ganzen Spitzenformen zu den Idealstufen der Hilbertschen Modulgruppe und die Abschätzung ihrer Fourierkoeffizienten. Acta Math. **92**, 309–345 (1954).

This paper carries over to the Hilbert modular group the methods and results of the Hecke-Petersson theory for functions of one variable.

The Hilbert modular group Γ is defined as follows: Let K be a totally real algebraic number field of finite degree n over the rational field. $\tau^{(1)},\cdots,\tau^{(n)}$ are n complex variables lying in the half-space $\operatorname{Im}\tau^{(k)}>0$. By a substitution $L\tau$ belonging to Γ we mean a set of n simultaneous linear fractional transformations

$$L\tau:\quad \tau^{(k)}\rightarrow\frac{a^{(k)}\tau^{(k)}+b^{(k)}}{c^{(k)}\tau^{(k)}+d^{(k)}}\quad(k=1, 2, \cdots, n),$$

where a, b, c, d are integers in K, and $\{a^k\}$ are the conjugates of a. $L\tau$ is associated with the matrix $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$, Γ is the group of all unimodular matrices, and $\Gamma(\mathfrak{c})$, the principal congruence subgroup of level $\mathfrak{c}$, is the subgroup of all unimodular matrices congruent elementwise to the identity matrix. Here $\mathfrak{c}$ is an integral ideal in K. Both Γ and $\Gamma(\mathfrak{c})$ are properly discontinuous and have fundamental regions with cusps lying on the boundary of the half-space $\operatorname{Im}\tau^{(k)}>0$.

The Hecke-Petersson method offers a recipe for the determination of the Fourier coefficients of an entire modular form of rational integral dimension $-r$ $(r\geqq 2)$. One subtracts from the given modular form a suitable linear combination of Eisenstein series (whose Fourier coefficients are known) so that the difference is an entire cusp form, i.e., a form vanishing at all the vertices of the fundamental region. By means of a completeness theorem, the discussion of the cusp forms is reduced to that of the Poincaré series. The latter are expanded in Fourier series, whose coefficients can be estimated non-trivially by using recent estimates of the Kloosterman sums developed by A. Weil [Proc. Nat. Acad. Sci. U. S. A. **34**, 204–207 (1948); MR **10**, 234]. The case $r=2$ is particularly difficult because the Poincaré series do not converge, and it is necessary to use a convergence factor method introduced by Hecke.

All these steps must now be carried out for the Hilbert modular group. The author first treats the case $r>2$. The required Eisenstein series have been discussed by Kloosterman [Abh. Math. Sem. Hamburg. Univ. **6**, 163–188 (1928); Math. Ann. **103**, 279–299 (1930)]. The completeness theorem for the Poincaré series was proved by Maass [Math. Ann. **117**, 538–578 (1940); MR **2**, 87]. The difficult reduction of the occurring Kloosterman sums to the usual sums is accomplished by the author (§4). The analytical details are very complicated.

The final result is: If $a^*(\nu)$ are the Fourier coefficients of an entire cusp form of rational integral dimension $-r(r>2)$ belonging to the group $\Gamma(\mathfrak{c})$, then

$$|a^*(\nu)|=O(N(\nu)^{\frac{1}{2}r-\frac{1}{2}+\epsilon}),$$

where $N(\nu)$ is the norm of ν. In §3 the author demonstrates that this estimate is correct also for $r=2$. *J. Lehner.*

★**Gundlach, Karl-Bernhard. Über die Darstellung der ganzen Spitzenformen zu den Idealstufen der Hilbertschen Modulgruppen und die Abschätzung ihrer Fourierkoeffizienten.** Dissertationen der Mathematisch-Naturwissenschaftlichen Fakultät der Westfälischen Wilhelms-Universität zu Münster in Referaten, Heft 5, pp. 5–6. Aschendorffsche Verlagsbuchhandlung, Münster, 1954. DM 3.50.

Summary of the paper reviewed above.

Lapin, A. I. The general duality law and a new foundation of the theory of class fields. Izv. Akad. Nauk SSSR. Ser. Mat. **18**, 335–378 (1954). (Russian)

Dans les constructions de la théorie globale des corps de classes, la loi de réciprocité pour les restes de puissances se déduit, en général, de la loi de réciprocité d'Artin, la théorie des corps de classes étant déjà préalablement construite. Or, il y a quelques années, I. R. Šafarevič [Mat. Sb. N.S. **26**(68), 113–146 (1950); MR **11**, 230] et, indépendemment, W. H. Mills [Amer. J. Math. **73**, 65–77 (1951); MR **12**, 592] ont résolu le problème de la "forme explicite de la loi de réciprocité," en donnant les définitions locales explicites du symbole des restes normiques $\left(\frac{\alpha,\mu}{\mathfrak{q}}\right)$ d'Hilbert. L'auteur, en se bornant au cas des extensions abéliennes à groupe de Galois de type $(p, p, \cdots, p)$ (où $p\neq 2$ est un nombre premier) d'un corps k de nombres algébriques, qui est de degré fini et contient les racines p-ièmes de l'unité, commence par établir directement la loi de réciprocité des restes de puissances p-ièmes et, ensuite, construit, en s'y basant, la théorie globale des corps de classes pour le cas considéré.

Le §1 du travail, de caractère préliminaire, est consacré à l'étude de certains groupes multiplicatifs de nombres et d'idéaux de k ou de son extension kummérienne $k(\mu^{1/p})$. Son contenu est analogue aux calculs qu'on fait habituellement pour établir les inégalités fondamentales, mais en s'intéressant non aux ordres ou indices, mais aux p-rangs des groupes considérés. Les démonstrations de ce paragraphe ne sont pas purement arithmétiques, car on se sert du théorème de Bauer. La plus grande partie du §2 peut être considérée comme démonstration, pour les extensions kummériennes $k(\mu^{1/p})/k$, d'une forme affaiblie de la loi de réciprocité

d'Artin, d'où, à la fin du paragraphe, on déduit, en se servant de la forme explicite de Šafarevič pour le symbole des restes normiques, la loi de réciprocité

$$\left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right)^{-1}=\prod_{\mathfrak{p}|p}\left(\frac{\beta,\alpha}{\mathfrak{p}}\right).$$

Ensuite, au §3, on démontre la validité, pour les extensions considérées, de la théorie des corps de classes sous la forme "idélienne" de Chevalley. D'abord, on prouve l'égalité, pour $k(\mu^{1/p})/k$, des groupes de Takagi et d'Artin et la formule de produit $\prod_{\mathfrak{q}}\left(\frac{\alpha,\mu}{\mathfrak{q}}\right)=1$ (où $\mathfrak{q}$ parcourt tous les idéaux premiers de k). Ceci étant et k^*, J_k désignant le groupe multiplicatif et le groupe des idèles de k, on définit un "produit scalaire (α,β) d'un $\alpha \in k^*$ et d'un $\beta=(\beta_{\mathfrak{q}}) \in J_k$, en posant $(\alpha,\beta)=\prod\left(\frac{\alpha,\beta_{\mathfrak{q}}}{k_{\mathfrak{q}}}\right)$, où $k_{\mathfrak{q}}$ est le corps $\mathfrak{q}$-adique de k. Si K/k est une extension kummérienne engendrée par les racines p-ièmes éléments de k, et si $\Gamma(K)$ désigne le groupe des éléments de k, qui sont des puissances p-ièmes des éléments de K, on montre que le sous-groupe de J_k orthogonal à $\Gamma(K)$ est précisément $P_k\cdot N_{K/k}J_k$, où P_k est le groupe des idèles principaux. [Remarque de référent: le travail fourmille d'erratas, en particulier de confusions de lettres romaines, gothiques, rondes, capitales, de lettres θ et ϑ, de signes $=$ et $\neq$ etc. Dans l'énoncé du lemme 5, il est imprimé "ramifié" au lieu de "non-ramifié," ce qui en renverse la conclusion. Egalement, certaines définitions sont mal rédigées, telle, p.ex., celle du genre principal, incompréhensible pour qui ne la connait pas déjà. Tout cela rend le travail illisible sauf pour les spécialistes au sens strict. Mais tous les énoncés (une fois purgés d'erratas) et toutes les démonstrations (du moins dans leur idée) sont corrects.]

M. Krasner (Paris).

Šafarevič, I. R. On the problem of imbedding fields. Izv. Akad. Nauk SSSR. Ser. Mat. **18**, 389–418 (1954). (Russian)

Le "problème d'immersion," qui fait l'objet du travail, est le suivant: soient G un groupe (d'ordre fini), g un sous-groupe invariant de G, F le quotient G/g. Si k_0 est un corps (de degré fini) de nombres algébriques, et si k/k_0 est une extension galoisienne, dont le groupe de Galois G_{k/k_0} est identifié avec F, il s'agit de prouver (si possible) l'existence et de donner une méthode de construction d'une surextension galoisienne K/k_0 de k/k_0, dont le groupe de Galois G_{K/k_0} puisse être identifié avec G de telle manière que l'application canonique de G_{K/k_0} sur G_{k/k_0} s'identifie avec l'homomorphisme canonique de G sur F. L'auteur résout ce problème [dont le cas où g est commutatif a été résolu par A. Scholz, Math. Z. **30**, 332–356 (1929)] dans le cas, où, à la fois: 1) G est le produit semi-direct $F\cdot g$ de F et de g; 2) g est un p-groupe; 3) ou bien la classe c de g est $<p$, ou bien l'ordre m de F est premier à p.

Au §1, certains types de groupes, appelés nilpotents dispositionnels et centralement dispositionnels, sont définis. Une puissance p^q de p étant fixée, soit $G_d^{(c)}$ le quotient du produit libre de d groupes cycliques $S_j=\{s_j\}$ d'ordre p^q par le c-ième terme de sa suite centrale descendante. Ce groupe est abélien si, et seulement si $c=1$ (en supposant $d>1$). Un groupe à opérateurs sera dit nilpotent dispositionnel par rapport à F et à p^q, si, en tant que groupe abstrait, il est un $G_{md}^{(c)}$ et si, dans les groupes cycliques S_j les générateurs peuvent être convenablement choisis et notés $t_{\xi,i}$ ($\xi \in F$; $i=1,2,\cdots,d$) de manière que le groupe d'opérateurs devienne F opérant selon les formules $t^{\sigma}_{\xi,i}=t_{\xi\sigma,i}$ ($\sigma \in F$). Ce groupe à opérateurs sera noté $G_{d,F}^{(c)}$, et tout p-groupe opéré par F est une image opératoirement homomorphe de quelque $G_{d,F}^{(c)}$. Un p-groupe opéré par F est dit centralement dispositionnel pour F si son centre est (pour quelque p^q) dispositionnel pour F, et on montre que tout p-groupe opéré par F et de classe $<p$ est une image opératoirement homomorphe de quelque groupe centralement dispositionnel. Soient Z le centre de $G_{d,F}^{(c)}$, $Z^{(r)}$ le groupe des puissances p^r-ièmes ($r\leqq q$) des éléments de Z, $G_{d,F}^{(c,r)}=G_{d,F}^{(c)}/Z^{(r)}$. En se servant d'un théorème de R. Thrall [Bull. Amer. Math. Soc. **47**, 303–308 (1941); MR **2**, 307] sur la structure du centre des $G_d^{(c)}$ quand $c<p$, l'auteur prouve que, sous cette hypothèse, $G_{d,F}^{(c,r)}$ est une image opératoirement homomorphe d'un groupe convenable Γ (opéré par F), où le noyau d'homomorphisme est un groupe abélien dispositionnel pour F et pour une puissance convenable de p. Ces préliminaires de la théorie des groupes permettent de réduire le problème d'immersion avec l'hypothèse $G=F\cdot g$ à une suite de problèmes d'immersion soumises aux hypothèses: 1) $F=\bar{F}\cdot h$, où $\bar{F}$ est un groupe donné, tandis que h est un p-groupe, qui dépend de pas de l'induction; 2) g est un groupe abélien dispositionnel pour $\bar{F}$ et pour $p=p^1$, isomorphe à $G_{1,\bar{F}}^{(1)}$ (pour $q=1$). On ne suppose pas, par contre, que G est un produit semi-direct de F et de g. Ce dernier problème se réduit au problème suivant de la théorie des corps: k_0, k, K étant des corps tels que $k_0\subset k\subset K$, que K/k_0 soit galoisienne avec $G_{K/k_0}=F=\bar{F}\cdot h$ et que $G_{K/k}=h$ (d'où $G_{k/k_0}=\bar{F}$), construire un corps $Q=K(\mu^{1/p})$ ($\mu \in K$), qui soit normal par rapport à k et dont tous les transformés par les $u \in G_{k/k_0}$ soient linéairement indépendants (dans leur ensemble) sur K, ce corps étant, en plus, tel que la construction analogue reste possible en remplaçant K par le composé K' des Q^u, où u parcourt G_{k/k_0}.

Le §2 est consacré à l'étude de ce problème. Une telle construction est sûrement possible si K/k est une extension scholzienne pour p^e, e étant suffisemment grand [autrement dit, le discriminant $D_{K/k}$ de K/k est premier à p, tous les facteurs premiers dans K de $D_{K/k}$ et de p et tous les idéaux à l'infini de K sont de degré absolu 1 et la norme absolue de tout facteur premier $\mathfrak{p}$ de $D_{K/k}$ dans k est $\equiv 1 \pmod{p^e}$] et si μ peut être choisi de manière que K'/k le soit aussi. Or, soit X une classe de p-invariance dans K/k [autrement dit, k^* étant le groupe multiplicatif de k et $K^{(p)}$ étant le groupe des puissances p-ièmes des éléments non-nuls de K, X est la classe (mod $k^*K^{(p)}$) d'un $\mu \in K$ tel que, pour tout $\tau \in G_{K/k}$, on a $\mu^{\tau}\equiv\mu$ (mod $K^{(p)}$). L'auteur avait montré, dans un travail précédent [Izv. Akad. Nauk SSSR. Ser. Mat. **18**, 261–296 (1954); MR **16**, 571] que la condition nécessaire et suffisante pour qu'on puisse choisir un $\mu \in X$ tel que $K(\mu^{1/p})/k$ reste scholzienne est l'égalité à 1 de certains invariants (χ, X) et $(X)_k$ qu'il y a défini. Cette condition est encore nécessaire, mais non suffisante pour que $\mu \in X$ puisse être choisi de manière que K'/k soit scholzienne. L'auteur définit d'autres invariants $[\chi, X]$ [qui comprennent les (χ, X) comme cas particulier, car $[\chi, X]=(\chi, X)$ si (χ, X) a un sens] et $[X]_{\mathfrak{p}}$ (qui sont trop compliqués pour qu'on puisse en donner la définition ici), et il montre, en se servant de la théorie des corps de classes, de la loi de réciprocité et des résultats de son travail des Izv. Akad. Nauk SSSR, Ser. Mat. **18**, 327–334 (1954) [MR **16**, 572], que μ peut être choisi ainsi si, et seulement si les $[\chi, X]$, $[X]_{\mathfrak{p}}$ et $(X)_k$ sont tous $=1$.

Au §3, l'auteur applique ce critère d'immersibilité et prouve, grâce à une théorie de composition d'homomorphismes d'une certaine forme, généralisant légèrement celle

développée dans Izv. Akad. Nauk SSSR. Ser. Mat. **18**, 261–296 (1954), que le problème d'immersion est résoluble dans les cas indiqués plus haut. Il considère, pour un k/k_0 donné de groupe F, tous les groupes $G_{d,r}^{(c,r)}$, et démontre, par induction sur r [avec c fixe et d variable; une telle induction en comporte aussi une par rapport à c avec r et d variables, car on a $G_{d,r}^{(c,q)}=G_{d,r}^{(c+1,0)}$], l'existence, pour tout d et pour tout $r\leqq q$, d'une surextension K/k_0 de k/k_0, dont le groupe soit, de la manière indiquée plus haut, $F\cdot G_{d,r}^{(c,r)}$. Il suffit, pour cela, de prouver l'existence, pour tout d, d'une surextension $\bar{K}/k_0$ de k/k_0, dont le groupe de Galois soit $F\cdot G_{d,r}^{(c,r-1)}$, qui soit scholzienne par rapport à k et dont toutes les classes X de p-invariance par rapport à k aient leurs invariants $[\chi, X]$, $[X]_p$ et $(X)_k$ égaux à 1. Or, si l'on suppose l'hypothèse de l'induction satisfaite, il existe, pour tout j, une extension $K_j/k_0\supseteq k/k_0$, dont le groupe de Galois est $F\cdot G_{j,r}^{(c,r-1)}$, et la théorie mentionnée de la composition des homomorphismes montre que, si j est suffisemment grand par rapport à d, K_j/k_0 contient une sous-extension $\bar{K}/k_0\supseteq k/k_0$ de groupe $F\cdot G_{d,r}^{(c,r-1)}$, dont toutes les classes de p-invariance par rapport à k ont leurs invariants $=1$.

M. Krasner (Paris).

Postnikov, A. G. On an application of the central-limit theorem of the theory of probability. Uspehi Mat. Nauk (N.S.) **10**, no. 1(63), 147–149 (1955). (Russian)

Let $r_1, \cdots, r_k$ be positive integers and $J_k(t)$ the number of integer solutions of $\sum_{i=1}^k x_i r_i^{-1}\leqq t$ in $|x_i|\leqq\frac{1}{2}r_i$ $(i=1, \cdots, k)$. Then $J_k(t)$, properly normalised, is asymptotically normal with error $O(k^{-1/2}\log k)$ according to Lyapunov's theorem. A lemma of T. Schneider [J. Reine Angew. Math. **175**, 182–192 (1936)] follows.

K. L. Chung.

Linnik, Yu. V. Application of the theory of Markov chains to the arithmetic of quaternions. Uspehi Mat. Nauk (N.S.) **9**, no. 4(62), 203–210 (1954). (Russian)

Let r be a positive odd integer and let there be σ_r distinct integral quaternions of norm r. The author considers products of the type $B=R_1\cdots R_s$ where the R_j are (not necessarily distinct) quaternions of norm r. The construction of such B which are primitive (do not contain a rational factor) follows a Markoff chain (in the sense of probability theory) since if B is primitive the R_{s+1} such that BR_{s+1} is primitive depend on R_s. In particular, let $\epsilon>0$ be fixed, R^* be some fixed quaternion of norm r, W be the number of primitive $B=R_1\cdots R_s$ and W^* the number of primitive $B=R_1\cdots R_s$ in which R^* occurs at most $(1-\epsilon)s/\sigma_r$ times; then $W^*/W<cr^{-\eta s}$, where c, η depend only on ϵ. From this can be deduced the following result of Malyšev which has application to the problem of the distribution of integral points on a sphere $x^2+y^2+z^2=m=$integer (*). Let $m\to\infty$ through the m for which (*) and $l_0^2+m\equiv0 \pmod r$ are soluble. Let L_α $(\alpha=1, \cdots, t(m))$ be the integral vectors such that $L_\alpha^2=-m$. Then the number $t^*(m)$ of α such that l_0+L_α is divisible on the left by a given R^* of norm r satisfies $t^*(m)/t(m)\to\sigma_r^{-1}$ as $m\to\infty$ [Yu. V. Linnik and A. V. Malyšev, Uspehi Mat. Nauk (N.S.) **8**, no. 5(57), 3–71 (1953); **10**, no. 1(63), 243–244 (1955); MR **16**, 450].

J. W. S. Cassels (Cambridge, England).

Cassels, J. W. S. Bounds for the least solutions of homogeneous quadratic equations. Proc. Cambridge Philos. Soc. **51**, 262–264 (1955).

Let $(a_1, a_2, \cdots, a_n)$ be a non-trivial solution in integers of the quadratic equation in $n\geqq2$ variables $\sum f_{ij}x_ix_j=0$, $1\leqq i<j\leqq n$, with integral coefficients f_{ij}, such that $\max|a_j|$ is a minimum. Let F denote $\max|f_{ij}|$. It is proved that

$$0<\max|a_j|\leqq\{\tfrac{1}{2}(3n^2+n-10)(n-1)^2F\}^{(n-1)/2},$$

by a method of descent. There appears to be no earlier work on such bounds for $n>3$; the author gives several references to papers on diagonal ternary forms.

I. Niven.

Kneser, Martin. Two remarks on extreme forms. Canad. J. Math. **7**, 145–149 (1955).

Let $f(x_1, \cdots, x_n)=\sum\sum a_{ik}x_ix_k$ be a positive definite quadratic form of determinant $D=\det(a_{ij})$, and let M be the minimum of f for integers $x_1, \cdots, x_n$ not all zero. Suppose the minimum is attained by s pairs of "minimal vectors" $\pm(m_{1k}, \cdots, m_{nk})$, so that $f(m_{1k}, \cdots, m_{nk})=M$ $(k=1, \cdots, s)$. The form is said to be extreme if M^n/D does not increase for any small variation of the coefficients a_{ij}. It is said to be perfect if it is determined by its minimal vectors. It is said to be eutactic if there exist s positive numbers ρ_k such that $\sum\rho_k(\sum m_{ik}x_i)^2$ is equal to the adjoint form. It was proved by Voronoï [J. Reine Angew. Math. **133**, 97–178 (1908)] that a form is extreme if and only if it is both perfect and eutactic. The author's first "remark" is a simpler proof for this theorem. The second is his announcement of a new extreme form in six variables:

$$\sum_1^3(x_j^2-x_jx_{j+3}+x^2_{j+3})+\left(\sum_1^6 x_i\right)^2,$$

for which $M=2$, $D=2^{-6}\,3^3\,13$ [cf. the following review].

H. S. M. Coxeter (Toronto, Ont.).

Barnes, E. S. Note on extreme forms. Canad. J. Math. **7**, 150–154 (1955).

The extreme forms [cf. the preceding review] were enumerated for $n\leqq5$ by Korkine and Zolotareff [Math. Ann. **11**, 242–292 (1877)], who also found three such forms for $n=6$. A fourth was added by Coxeter [Canad. J. Math. **3**, 391–441 (1951), p. 439; MR **13**, 443]. A fifth was discovered simultaneously by M. Kneser and the author. In a sense, they were very nearly anticipated by N. Hofreiter [Monatsh. Math. Phys. **40**, 129–152 (1933)], whose incorrect form has determinant $2^{-6}\,3^3\,53$; we merely have to replace his 53 by 52 to obtain D for the new form.

The author also answers the reviewer's challenge [loc. cit., p. 393] to find a perfect form that is not extreme (and therefore not eutactic). One such form (in eleven variables) is

$$\sum_1^3(x_j^2-x_jx_{j+3}+x^2_{j+3})+\left(\sum_1^{11} x_i\right)^2+\sum_7^{11}x_k^2.$$

H. S. M. Coxeter (Toronto, Ont.).

Barnes, E. S. The non-negative values of quadratic forms. Proc. London Math. Soc. (3) **5**, 185–196 (1955).

Let $Q(x_1, x_2, \cdots, x_n)$ be an indefinite quadratic form in n integral variables, of determinant $d\neq0$ and signature s. Define $k_{n,s}$, depending only on n and s, as the least constant for which the inequalities (*) $0\leqq Q(x_1, \cdots, x_n)\leqq(k_{n,s}|d|)^{1/n}$ always admit integral solutions $x_1, x_2, \cdots, x_n$, not all zero. For binary forms the results are classical and for $n\geqq5$ it is conjectured that Q takes arbitrarily small values; hence, only the cases $n=3$ and $n=4$ are of interest. Oppenheim [same Proc. (3) **3**, 328–337, 417–429 (1953); MR **15**, 291, 607] solved the related problem with strict inequality on the left and with the exclusion of $i-1$ classes of zero forms $Q_{n,\nu}$ $(\nu=1, 2, \cdots, i-1)$. Oppenheim's constants $c_{n,s}^{(i+1)}$ are,

therefore, upper bounds for the $k_{n,s}$. The author improves these bounds and obtains: $k_{3,1}=4/3$ (excluding the class equivalent to $-x^2+8(y^2+yz+z^2)$, the constant can be reduced to <1.332), $k_{3,-1}\leqq 4$; $k_{4,0}\leqq 64/81$, $k_{4,2}\leqq 32/27$, $k_{4,-2}\leqq 64/27$. A note adds the further improvements, due to Oppenheim, that $k_{4,0}=64/81$ and $k_{3,-1}=16/5$. All these results are obtained by comparatively simple arguments from the following key lemma (Theorem 3 of the paper): Let $\phi(x, y)$ be a non-zero, indefinite binary quadratic form of determinant $D>0$. Then ϕ represents, for integral x, y, values $p>0$, $-n<0$, satisfying $p^2n\leqq D^2/16$. If ϕ is not equivalent to a positive multiple of $-x^2+8y^2$, then the inequality may be replaced by $p^2n<D^2/16.07$. The proof of the lemma is based on the theory of continued fractions.

E. Grosswald (Philadelphia, Pa.).

Tornheim, Leonard. Asymmetric minima of quadratic forms and asymmetric Diophantine approximation. Duke Math. J. 22, 287–294 (1955).

Let $f(x, y)=ax^2+bxy+cy^2$ be an indefinite quadratic form not representing 0 and normalized to $b^2-4ac=1$. Let p^{-1}, n^{-1} be respectively the infima of the positive values of $f(x, y)$, $-f(x, y)$ for integral x, y and put $B=B(f)=\max(p^2, k^2n^2)$, where $k\geqq 1$ is fixed. Segre [same J. 12, 337–365 (1945); MR 6, 258] has shown that $B\geqq B_1=k^2+4k$ with equality only when k is an integer and f is equivalent to $x^2-kxy-ky^2$. The author shows that for $k\geqq 2$, k an integer, the next highest value is

$$B_2=\left[\frac{k^2+k+(3k+1)(k^2+4k)^{1/2}}{2(2k-1)}\right]^2,$$

taken only for forms equivalent to

$$x^2-(k-1)xy-\tfrac{1}{2}[3k-(k^2+4k)^{1/2}]y^2.$$

But B_2 is a limit point on the right of $B(f)$ values, so there is no B_3, in striking contrast to the well-known Markoff chain for $k=1$. The author investigates in more detail the $B(f)$ values in a neighbourhood on the right of B_2. He also proves the analogous results for diophantine approximation. The proof shows first that if $B(f)$ is fairly small the continued fraction associated with f has the shape $(\cdots, 1, X_1, 1, X_2, 1, X_3, \cdots)$ where $X_j=k$ or $k-1$ and then considers the structure of the X_j in detail. [For non-integral k see Barnes and Swinnerton-Dyer, Acta Math. 92, 199–234 (1954); MR 16, 802.]

J. W. S. Cassels.

Mahler, Kurt. On a problem in Diophantine approximations. Arch. Math. 6, 208–214 (1955).

Let $L(x)$ be a linear form in n variables $x_1, \cdots, x_n$; assume that $n\geqq 2$, and that the coefficients of this form are real numbers of absolute value not greater than $a\geqq 1$. Denote by v_{n-1} the volume of the $(n-1)$-dimensional polyhedron defined by the inequalities

$$\max(|x_i|)\leqq 1, \quad |x_1+\cdots+x_n|\leqq 1.$$

Then for every real number $N\geqq 1$, there are integers $x_1, \cdots, x_n$ not all zero such that $|L(x)|\leqq N^{-1}$ and

$$\max(|x_i|)\leqq 2(aN/v_{n-1})^{1/(n-1)}.$$

This is an improvement of the usual Dirichlet theorem, in which the right-hand side of the last inequality is $2(anN)^{1/(n-1)}$, since $v_{n-1}\geqq 2$ and $v_{n-1}\sim(3/2\pi n)^{1/2}2^n$ as $n\rightarrow\infty$. It is obtained by applying Minkowski's theorem on convex bodies, rather than the Schubfachprinzip.

W. J. LeVeque (Ann Arbor, Mich.).

***de Vries, Dirk. Metrische onderzoekingen van Diophantische benaderingsproblemen in het niet-lacunaire geval. [Metrical investigations of Diophantine approximation-problems in the non-lacunary case.]** Thesis, Free University of Amsterdam, 1955. 93 pp.

A well known theorem of A. Hinčin [Math. Z. 24, 706–714 (1926)] asserts that if $\psi(x)$ is positive and continuous and $x\psi(x)$ tends monotonically to zero as $x\rightarrow\infty$, then the inequality $|x\theta-y|<\psi(x)$ has infinitely many integral solutions x, y $(x>0)$ for almost all or almost no θ, according as $\sum_1^N\psi(x)$ diverges or converges. J. W. S. Cassels [Proc. Cambridge Philos. Soc. 46, 209–218 (1950); MR 12, 162] generalized this theorem, requiring only that $\psi(x)$ be a positive decreasing function of the positive integral variable x, and replacing the above inequality by $|f(x)\theta-y|<\psi(x)$, where $f(x)$ is a Σ-function, i.e., a positive increasing integral-valued function with the following property: if μ_z is the number of fractions $j/f(x)$ $(0<j<f(x))$ which cannot be simplified to the form $k/f(z)$ $(z<x)$, then

$$\liminf_{N\rightarrow\infty} N^{-1}\sum_{z=1}^{N}\frac{\mu_z}{f(x)}>0.$$

Up to now, the only non-lacunary functions known to be Σ-functions were the powers x^k, k a positive integer. The first part of this thesis is devoted to extending the list, so that it now includes (among others) polynomials with integral coefficients, $[P_1(x)]$, $[P_1(x)/P_2(x)]$, $[(P_3(x))^{1/n}]$, and $[\log^d x]$, where P_1, P_2, P_3 are polynomials with rational coefficients and $n|\deg P_3$. Second, it is shown that if $\psi(x)$ is positive and decreasing, and $x\psi(x)$ is monotone, and if $a>0$ and b are rational, then the inequality

$$0\leqq(ax+b)^{1/n}\theta-y<\psi(x)$$

has infinitely many integral solutions for almost all or almost no θ, according as $\sum\psi(x)$ diverges or converges. Cassels' theorem is not applicable here, since $(ax+b)^{1/n}$ is not integral-valued. Finally, V. Jarník's work [Math. Z. 33, 505–543 (1931)], on the Hausdorff dimension of the exceptional θ-set when $\sum\psi(x)$ converges, is extended; it is shown for example that if $P(x)$ is a polynomial of degree n with integral coefficients, and if $\alpha>n+1$ is constant, then $\dim V=(n+1)/\alpha$, where V is the set of numbers θ for which the inequality $|\theta-y/P(x)|<x^{-\alpha}$ has infinitely many solutions.

All of these results depend on evaluating limits of sums of the form $N^{-1}\sum_1^N\phi(f(x))/f(x)$, where φ is the Euler function. If such a limit is positive, the inequality occurring above in the definition of Σ-function follows from the fact that $\mu_x\geqq\varphi(x)$.

W. J. LeVeque (Ann Arbor, Mich.).

Have

***Lutz, Elisabeth. Sur les approximations diophantiennes linéaires P-adiques.** Actualités Sci. Ind., no. 1224. Hermann & Cie, Paris, 1955. 106 pp. 1200 francs.

This thesis constitutes an extensive investigation of approximation questions concerning a system Λ of p linearly independent linear forms $L_j(x)$ with P-adic coefficients, in n rational integral variables $(x_1, \cdots, x_n)=x$. (Λ is said to be of signature (p, n).) Certain of the results proved here were announced earlier [C. R. Acad. Sci. Paris 232, 587–589, 667–669, 784–786, 1389–1392 (1951); MR 13, 116, 117]. The treatment is self-contained, except for general topological and measure-theoretic background

Define P: a rational prime; Q: the rational numbers; Q_P: the P-adic completion of Q; Q_P^n: the n-dimensional vectors with components in Q_P; E_P^n: the vectors of Q_P^n with P-adic

integral components; Z^n: the vectors of $Q_P{}^n$ whose components are rational integers. If Λ is of signature (p, n), it is equivalent to a canonical system L, in which $L_j(x)=x_j+\sum_{i=1}^q a_{ij}x_{p+i}$ $(j=1, \cdots, p)$, where $p+q=n$. Λ is said to be free if the a_{ij} are algebraically independent. The system M, in which $M_i(y)=y_{p+i}-\sum_{j=1}^p a_{ij}y_j$ for $i=1, \cdots, q$, is called the associate of L. For $x \in Z^n$, put

$$\Lambda(x)=\max_{1\leqq j\leqq p} (|\Lambda_j(x)|_P) \quad\text{and}\quad H(x)=\max_{1\leqq i\leqq n} (|x_i|).$$

x is said to be primitive if $\gcd(x_1, \cdots, x_n)=1$. Define the rational integer $\delta(\Lambda)$ by the relation $P^{-\delta}=\max_D (|\det D|_P)$, where D ranges over the minors of rank p in the matrix of coefficients of Λ. $\rho(\Lambda)$ has a similar but more complicated definition. Finally, if $\varphi(\lambda)$ has non-negative value for positive rational integral λ, and $x^{(0)} \in E_P{}^n$, the notation $\Lambda_{x^{(0)}} \dashv\dashv \varphi(\lambda)$ means that there is a positive constant c such that for all sufficiently large λ there is a solution $x \in Z^n$ of the inequalities $\Lambda(x+x^{(0)})\leqq P^{-\lambda}$ and $H(x)c\leqq \varphi(\lambda)$. In case $x^{(0)}=0$, the restriction $x\neq 0$ is also imposed, and we write $\Lambda \dashv\dashv \varphi(\lambda)$.

The following are among the principal theorems: I. Let $\lambda_1, \cdots, \lambda_p$ be rational integers $\geqq \rho(\Lambda)$, and let c be a positive constant. Then there exists $x \in Z^n$, primitive except for a power of P, such that $|\Lambda_j(x)|_P\leqq P^{-\lambda_j}$ $(j=1, \cdots, p)$ and $0<H(x)\leqq c$, if $c^n\geqq P^{\lambda_0-\delta(\Lambda)}$, where $\lambda_0=\sum_1^p \lambda_j$. II. If $\Lambda(x)$ is never 0 for $x \in Z^n$, $x\neq 0$, the inequality $H^n(x)\Lambda^p(x)\leqq P^{-\delta}$ has infinitely many primitive solutions x. III. If $\varphi(\lambda)$ is as described above, and $1\leqq p\leqq \frac{1}{2}n$, there exists a free canonical system L of signature (p, n) such that $L \dashv\dashv \varphi(\lambda)$, and these systems are everywhere dense in the space $Q_P{}^{pq}$ of coefficients $(a_{11}, \cdots, a_{qp})$. IV. Given a canonical system L of signature (p, n) $(1\leqq p<n)$ and an $x^{(0)} \in E_P{}^n$, there exists a function $\epsilon(\lambda)$ tending to zero with λ^{-1} such that $L_{x^{(0)}} \dashv\dashv \epsilon(\lambda)P^{\lambda}$, if and only if $\sum_1^p L_j(x^{(0)})y_j$ is a rational integer for every $y \in Z^n$ such that $M(y)=0$. V. For every system L having integral P-adic coefficients, there is an $x^{(0)} \in E_P{}^n$ such that the inequality $L(x+x^{(0)})\leqq (nP^{\delta}H(x))^{-n/p}$ has no solution $x\neq 0$. VI. Let $f(l)$ be a sufficiently regular function, defined and positive for $l>0$. Let the forms of the system L have integral coefficients in Q_P. Then the number of solutions x of the inequality $L(x)\leqq f(H(x))$ is finite or infinite for almost all (Haar measure) $a=(a_{11}, \cdots, a_{qp}) \in E_P{}^{pq}$, according as the series $\sum_1^\infty h^{n-1}f^p(h)$ converges or diverges.

W. J. LeVeque (Ann Arbor, Mich.).

Schneider, Theodor. Über die Irrationalität von π. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1954**, 99–101 (1955).

By use of a cleverly designed interpolation series for the exponential function, the author gives a brief proof that, unless $b=0$, e^b and b cannot both belong to the field $R(i)$, the rational numbers with i adjoined. The irrationality of π is seen to be a consequence by taking $b=i\pi$. *I. Niven.*

ANALYSIS

Drazin, M. P. Some inequalities arising from a generalized mean value theorem. Amer. Math. Monthly **62**, 226–232 (1955).

The theorem figuring in the title is: If $f(x)$ is continuous for $a\leqq x\leqq b$ and $f_{(x)}^{(n)}$ exists for $a<x<b$, then to every sequence $h_i>0$ $(i=1, 2, \cdots, n)$ there can be found a set of functions $\xi_k(x)$ $(k=1, 2, \cdots, n)$ defined respectively for $a\leqq x\leqq b-h_1-\cdots-h_k$ such that $0<\xi_k(x)<h_1+\cdots+h_k$ and $\Delta_{h_k}\Delta_{h_{k-1}}\cdots\Delta_{h_1}f(x)=f^{(k)}[x+\xi_k(x)]$, where

$$\Delta_h f(x)=[f(x+h)-f(x)]/h$$

[cf. T. Popoviciu, Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 4, 353–356 (1954), esp. p. 356; MR 16, 452]. From this and from the identity

$$\sum_{l=0}^n \binom{n}{l} y^l a_l=(-1)^n\sum_{m=0}^n \binom{n}{m}(-1-y)^m\Delta^{n-m}a_m$$
$$[\Delta a_l=a_{l+1}-a_l,\ \Delta^{k+1}a_l=\Delta(\Delta^k a_l)]$$

the author derives the inequalities $\sum_{l=0}^n \binom{n}{l}y^l a_l>0$ if $y\geqq -1$, $(-1)^n\Delta^n a_0>0$, $(-1)^{n-m}\Delta^{n-m}a_m\geqq 0$ $(m=1, 2, \cdots, n)$ and $(-1)^n\sum_{l=0}^n \binom{n}{l}y^l a_l>0$ if $y\leqq -1$, $\Delta^n a_0>0$, $\Delta^{n-m}a_m\geqq 0$, further $\sum_{l=0}^n \binom{n}{l}y^l\prod_{p=1}^q f_p(l)>0$, if $y>-1$ and $f_i(x)$ $(i=1, 2, \cdots, n)$ is continuous for $0\leqq x\leqq n$, $f_i^{(n)}(x)$ exists for $0<x<n$, $f_i(n)>0$ and

$$(-1)^k f_p^{(k)}(x)\geqq 0$$
$$(p=1, 2, \cdots, q;\ k=1, 2, \cdots, n;\ n-k<x<n).$$

Also, a combination of the identity and the generalized mean-value theorem above, a generalization of this identity, and a remark on the nature of the set of mean-value points in the ordinary mean-value theorem are given.

J. Aczél (Debrecen).

Biernacki, Mieczysław. Sur des inégalités remplies par des expressions dont les termes ont des signes alternés. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **7** (1953), 89–102 (1954). (Polish and Russian summaries)

The study of inequalities involving positive terms with alternating signs is now continued, both in the spirit of Weinberger [Proc. Nat. Acad. Sci. U. S. A. **38**, 611–613 (1952); MR **14**, 24], who established a particular algebraic inequality of this sort, and in the spirit of Bellman [Amer. Math. Monthly **60**, 402 (1953); MR **14**, 957], who showed Weinberger's inequality to be an instance of a general inequality concerning convex functions. Thus the author shows first that, while certain classical inequalities do not remain valid for all sets of positive terms with alternating signs, other classical inequalities, including Chebyshev's, do remain valid in this case. Secondly, he establishes the following generalization of Bellman's result: If $\phi(u)$ is convex for $u\geqq 0$, and if $0<a_1\leqq b_1\leqq a_2\leqq\cdots\leqq b_{n-1}\leqq a_n$ and $0\leqq b_n$, $b_1+\cdots+b_n\leqq a_1+\cdots+a_n$, then

$$\phi(a_1)+\cdots+\phi(a_n)-\phi(b_1)-\cdots-\phi(b_n)$$
$$\geqq\phi(a_1+\cdots+a_n-b_1-\cdots-b_n)-\phi(0).$$

We note that a consideration of the geometry involved can lead to further generalizations in the same direction.

E. F. Beckenbach (Los Angeles, Calif.).

Tatarkiewicz, Krzysztof. Sur une inégalité intégrale. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **7** (1953), 83–87 (1954). (Polish and Russian summaries)

The inequality referred to in the title is the following: If A_1 and A_2 are two measurable sets, f is a measurable and g a summable function on $A=A_1+A_2$, and

$$0\leqq f(p_1)\leqq g(p_1)\leqq g(p_2)\leqq f(p_2)$$

for every $p_1 \varepsilon A_1$, $p_2 \varepsilon A_2$, then $\int_A g(p)dp \leqq \int_A f(p)dp$ implies $\int_A g(p)^2 dp \leqq \int_A f(p)^2 dp$, where also the values ∞ for the integrals (of course, except for the integral $\int_A g(p)dp$) are allowed. A counter-example shows that measurability of g instead of summability does not suffice. The most simple particular case of the inequality above states that if $f(x)$, $g(x)$ are non-negative and non-decreasing for $a \leqq x \leqq b$, further $f(x) \leqq g(x)$ for $a \leqq x < d$, while $f(x) \geqq g(x)$ for $d < x \leqq b$, then $\int_a^b g(x)dx \leqq \int_a^b f(x)dx$ implies

$$\int_a^b g(x)^2 dx \leqq \int_a^b f(x)^2 dx.$$

Also, an application to a problem of the calculus of variations is shown. *J. Aczél* (Debrecen).

Anderson, T. W. The integral of a symmetric unimodal function over a symmetric convex set and some probability inequalities. Proc. Amer. Math. Soc. **6**, 170–176 (1955).

Let f be a function on Euclidean n-space which is non-negative, symmetric with respect to the origin, and such that the set of points for which it is at least equal to a given value is convex. Then for a given point y and convex set E the integral of $f(x+ky)$ for x over E is at least equal to that of $f(x)$. The proof depends on the Brunn-Minkowski inequality for the volumes of convex bodies. It follows that the probability of a random vector with density f falling in a given symmetric convex set is at least equal to the corresponding probability for the sum of this vector and another independent one. The results are extended to obtain inequalities on the distribution of functionals of stochastic processes. *K. L. Chung* (Syracuse, N. Y.).

Bielecki, A. On an elementary method of proof of the Gauss-Ostrogradskiĭ theorem. Wiadom. Mat. (2) **1**, 112–121 (1955). (Polish)

Obrechkoff, N. Sur les zéros de quelques fonctions rationnelles et réelles. C. R. Acad. Bulgare Sci. **7** (1954), no. 2, 1–4 (1955). (Russian. French summary)

Let

$$F(x) = \sum_{i=1}^{m} \frac{A_i}{x-a_i} + \sum_{j=1}^{p} B_j\left(\frac{1}{x-c_j}+\frac{1}{x-\bar{c}_j}\right),$$

where $A_i>0$, $B_j>0$, $a_1<a_2<\cdots<a_m$, and c_j is not real. Set

$$\eta = a_k + \frac{A'}{A'+A_{k+1}}(a_{k+1}-a_k), \quad A' = \sum_{i=1}^{k} A_i + 2\sum_{j=1}^{p} B_j.$$

It is proved that the interval $a_k < x \leqq \eta$ contains an odd number of zeros of $F(x)$, if the open circle with diameter end points a_k and η contains no c_j. Two other theorems on the zeros of $F(x)$ are proved. *A. W. Goodman.*

Čakalov, Lyubomir. General quadrature formulas of Gaussian type. Bŭlgar. Akad. Nauk. Izv. Mat. Inst. **1**, no. 2, 67–84 (1954). (Bulgarian. Russian summary)

In the approximate formula

$$(*) \qquad \int_a^b f(x)dx = \sum_{k=1}^{m} \sum_{i=0}^{r_k} A_{ki} f^{(i)}(a_k)$$

the author seeks to determine the real coefficients A_{ki} and real abscissas a_k in such a way that (*) be exact for all polynomials of degree less than N, where N is as large as possible. He shows that this maximal N is $\sum_{k=1}^{m} (2[r_k/2]+2)$ and gives a method to find the a_k and the A_{ki}. The special case $r_1 = \cdots = r_k = 2s$ has been treated by Turán [Acta Sci. Math. Szeged **12**, Pars A, 30–37 (1950); MR **12**, 164]. *G. G. Lorentz* (Detroit, Mich.).

Kreĭn, M. G., and Rehtman, P. G. Development in a new direction of the Čebyšev-Markov theory of limiting values of integrals. Uspehi Mat. Nauk (N.S.) **10**, no. 1(63), 67–78 (1955). (Russian)

This is a continuation of Kreĭn's paper [Uspehi Mat. Nauk (N.S.) **6**, no. 4(44), 3–120 (1951); MR **13**, 445] in which, among other things, the theory of absolutely monotonic functions on an infinite interval was deduced from the theory mentioned in the title, which deals with questions connected with moment problems. Here the general moment problem $c_k = \int_E u_k(t)d\sigma(t)$ is dealt with when the set E consists of a point set with a single limit point. Markov's theorem on the maximum and minimum of $\int_E \Omega(t)d\sigma(t)$ is correspondingly extended. As applications, the authors deduce theorems, some old and some new, on absolutely monotonic functions on a finite interval. *R. P. Boas, Jr.* (Evanston, Ill.).

Theory of Sets, Theory of Functions of Real Variables

***Littlewood, J. E. The elements of the theory of real functions. 3d ed.** Dover Publications, Inc., New York, 1955. vii+71 pp. Paper, \$1.35; cloth, \$2.85.

Although the general arrangement and character remain the same, this edition contains some extensive revisions of the second edition [Heffer, Cambridge, 1926].

***Frolov, N. A. Teoriya funkciĭ deĭstvitel'nogo peremennogo. [Theory of functions of a real variable.]** Gosudarstv. Učebno-Pedagog. Izdat., Moscow, 1953. 164 pp. 3.60 rubles.

A textbook of introductory character. The chapter headings are: General theory of sets; The set of real numbers; The theory of point sets; Functions; Continuous curves; Measurement of sets; Riemann integral; Lebesque integral; Role of Soviet mathematicians in the development of the theory of functions of a real variable.

Fraïssé, Roland. Sur quelques classifications des systèmes de relations. Publ. Sci. Univ. Alger. Sér. A. **1** (1954), 35–182 (1955).

This paper appeared earlier as the author's thesis [Univ. de Paris, 1953] and was reviewed in MR **15**, 296.

Fraïssé, Roland. Sur certains opérateurs dans les classes de relations. C. R. Acad. Sci. Paris **240**, 2109–2110 (1955).

Une m-relation de base E est chaque transformation R de E^m sur $\{+,-\}$; pour $m=0$, R désigne soit $(E,+)$ soit $(E,-)$ et est dénotée $|R|$. L'isomorphisme d'une restriction φ de R sur une restriction de S s'appelle un isomorphisme restreint (i.r.) de R vers S. On considère aussi la relation φ_0 de base vide; φ_0 est un isomorphisme restreint de R vers S si $m>0$ ou si $|R|=|S|$, $m=0$. L'A. classifie les i.r. de R vers S au moyen de γ-parenté, $\gamma=\{^n_p\}$; R et S sont γ-parentés, $R \overset{\gamma}{\sim} S$ ou $R \overset{n}{\underset{p}{\sim}} S$, si φ_0 est un isomorphisme de R vers S. Si à chaque m-relation R on associe une m'-relation $P(R)$ de même base, l'A. dit que P est un opérateur d'espèce $(m \to m')$. P est γ-opérateur si chaque γ-isomorphisme d'une m-relation vers S est un i.r. de $P(R)$ vers $P(S)$. L'A. énonce

quelques propriétés des opérateurs, par exemple: m, m' étant donnés, les γ-opérateurs d'espèce $(m\to m')$ sont en nombre fini. On a $R\overset{\gamma}{\sim}S$ si et seulement si $|P(R)|=|P(S)|$ pour chaque γ-opérateur P d'espèce $(m\to 0)$.

G. Kurepa (Zagreb).

Fraïssé, Roland. La construction des γ-opérateurs et leur application au calcul logique du premier ordre. C. R. Acad. Sci. Paris **240**, 2191–2193 (1955).

Suite de la Note analysée ci-dessus. A partir des γ-opérateurs simple on peut construire chaque γ-opérateur. Ce procédé de construction conduit à associer d'une certaine façon aux énoncés P en algèbre de relations les énoncés du calcul logique du premier ordre de telle sorte que chaque formule de celui-ci, ayant un seul prédicat, représente un γ-opérateur. *G. Kurepa* (Zagreb).

Fraïssé, Roland. Sur l'extension aux relations de quelques propriétés des ordres. Ann. Sci. Ecole Norm. Sup. (3) **71**, 363–388 (1954).

Proof of propositions announced earlier [C. R. Acad. Sci. Paris **237**, 508–510, 540–542 (1953); MR **15**, 192]. The results are specified for order relations. Four problems are considered. E.g. what is the power of the set of all the Γ satisfying D (resp. $\bar{D}$)? The definition of D (resp. $\bar{D}$) follows. Let K be any class of relations such that if $X \in K$ then also $Y \in K$ for each isomorph Y of X. Let A, B be elements of K with finite bases F, G respectively. Condition D: If A, B have a common restriction X, there exist a $C \in K$ and two isomorphisms φ, ψ of A, B onto restrictions of C so that $\varphi(x)=\psi(x)$, provided x belongs to the basis of X. Condition $\bar{D}$. If the restrictions of A, B on $F\cap G$ are equal, K contains an extension of both A, B. The author remarks that for every homogeneous relation R, the class Γ_R satisfies D [notations as in MR **15**, 192]. *G. Kurepa* (Zagreb).

Kurepa, G. Sur les fonctions réelles dans la famille des ensembles bien ordonnés de nombres rationnels. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. (N.S.) **12**, 35–42 (1954).

Let $(S, <)$ be a simply ordered set. Let ωS denote the family of all non-empty well ordered subsets of S, ordered in the following manner: For x and y in ωS, $x<_iy$ if and only if x is a proper initial section of y. Consider the existence of a function f from S into ωS such that $f(x)<f(y)$ if $x<_iy$. The main result is that if S is the set of rational numbers (denoted by R), then no such function f exists. It is proved by an application of the ensuing proposition: The ordered set $(R, <_i)$ is the union of $\aleph_1$ of its antichains. The cardinal $\aleph_1$ cannot be lowered. *S. Ginsburg* (Hawthorne, Calif.).

Kurepa, Đuro. On real functions on the family of ordered sets of rational numbers. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **296**, 85–93 (1953). (Serbo-Croatian)

Serbo-Croatian version of the paper reviewed above.

Kurepa, G. Über die Faktoriellen endlicher und unendlicher Zahlen. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. (N.S.) **12**, 51–64 (1954).

Attempts are made to find meanings for $n!$ if n is a transfinite number. The author defines $n!$ as the number of permutations of E, where E has power n. For each transfinite number n, $n!=2^n$. If n is finite and >1, then $n!=2^n$ is never true. For any natural number n let $I(n)$ be the set of natural numbers $<n$ and $P(n)$ the set of functions f from $I(n)$ into $I(n)$ such that $0\leqq f(\xi)\leqq\xi$ for each ξ in $I(n)$. After observing that the power of $P(n)$ is $n!$, he considers what occurs when n is infinite. Finally he shows that each permutation group can be represented by square dyadic (that is each term is either 0 or 1) matrices which have exactly one 1 in each row and column (possibly infinite number).

S. Ginsburg (Hawthorne, Calif.).

Kurepa, Đuro. On factorials of finite and infinite numbers. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **296**, 105–122 (1953). (Serbo-Croatian)

Serbo-Croatian version of the paper reviewed above.

Kurepa, Georges. Sur une classe de continus ordonnés. C. R. Acad. Sci. Paris **240**, 2283–2284 (1955).

For any ordinal number $\alpha>0$ let $2(\alpha)$ be the set of all functions from the set of ordinals $<\alpha$, $I(\alpha)$, to the set $\{0, 1\}$; ordered by first differences. Let $2[I(\alpha)]$ be the same set, partially ordered by: $g\leqq h$ if and only if $g(\xi)\leqq h(\xi)$ for all ξ. ($2(\alpha)$ is a linear extension of $2[I(\alpha)]$). Let $PI(\alpha)$ be the family of all subsets of $I(\alpha)$, ordered by set inclusion. The following results are obtained: (1) For X in $PI(\alpha)$, let $\phi(X)=\phi_X$, ϕ_X being the characteristic function of X. Then ϕ is an isomorphism of $PI(\alpha)$ onto $2[I(\alpha)]$. The consideration of ϕ^{-1} as an isomorphism of $2(\alpha)$ onto $PI(\alpha)$ introduces a new order $<'$ into $PI(\alpha)$. Then for X and Y in $PI(\alpha)$, $X<'Y$ if and only if the first element of $(X-Y)\cup(Y-X)$ belongs to Y. (2) For $2\leqq m$, $n<\omega$, $m(\omega_\alpha)$ and $n(\omega_\alpha)$ are isomorphic. For $2\leqq m$, $n\leqq\omega_\alpha$, $tm(\omega_\alpha)\leqq tn(\omega_\alpha)$ and $tn(\omega_\alpha)\leqq tm(\omega_\alpha)$, where tE denotes the order type of E. The author, in a later paper, intends to identify consecutive points in the $m(\omega_\alpha)$ and obtain ordered continua. No proofs are given. *S. Ginsburg* (Hawthorne, Calif.).

Popadić, Milan S. On inductive systems. Fac. Philos. Univ. Skopje. Sect. Sci. Nat. Annuaire **7**, no. 1, 65 pp. (1954). (Serbo-Croatian. English summary)

The Serbo-Croatian text (pp. 1–55) of the paper is the same as the author's thesis presented at the beginning of 1954 (the reviewer was the mentor); the English text presents some deviations from it. The paper deals with general considerations on induction, i.e. on exhaustion of a set by means of the elements of a system of sets. The paper is connected with some of the reviewer's works (only partially published).

The main aim of the author is to study inductive systems as systems of subsets of a set M by means of which, according to a procedure π, one can exhaust M. Roughly speaking, π consists in associating to each considered $D\subset M$ a larger part $fD\supset D$. A system $SM\subseteq PM$ is inductive relative to M if for every set N the relation $M\subseteq N$ is implied by the following two conditions: 1) $(S(M)-\{\Lambda\})\cap PN$ is non vacuous; 2) for every element B of $(S(M)-\{\Lambda, M\})\cap PN$ there exists a $C\in SM\cap PN$ so that $B\subset C$. Then one has this "fundamental theorem" (first formulation): In order that a $SM\subseteq PM$ covering M be inductive relative to M it is necessary and sufficient that for every $D\subset M$ the system $SM\cap PD$, if it be nonvoid, contain a last element (Th.3.3.1). PX denotes the system of all subsets of X; PX, as well as subsets of PX, is meant to be ordered by means of $\subseteq$. In order that for a chain C the system of its closed intervals [resp. elementary sections] be inductive, it is necessary and sufficient that both S and the dual S^* be well ordered [resp. that C have no gap]. For a chain C, let C^n denote the cardinal ordering of $C\times C\times\cdots\times C$ (n times). For

any $x=(x_1, \cdots, x_n) \in C^n$ let $(-, x|_{C^n}$ denote either the set $\{y|y \in C^n, y \leqq x\}$ or the set

$$\{(y_1 \cdots y_n)|y_1<x_1, \cdots, y_n<x_n, (y_1 \cdots y_n) \in C^n\}.$$

In order that the system of sets $(-, x|_{C^n}$ $(x \in C^n)$ be inductive for C^n, it is necessary and sufficient that the chain C have no interior gap (Th. 8.2.2; this theorem answers a question of the reviewer's).

The author gives two still more general formulations of induction procedures in terms of binary relations and mappings, respectively. Let φ be a binary relation in M, i.e., $\varphi \subseteq M^2$. Let $pr_1\varphi$ (resp., $pr_2\varphi$) be the set of the first (resp., second) components of elements of φ. Let M, N be any sets. A system $SM \subseteq PM$ is inductive for M relative to an "inductor", (S_1M, φ), where $S_1M \subseteq PM$ and φ is a binary relation with $pr_1\varphi = SM$, $pr_2\varphi = PM$, if the relation $M \subseteq N$ is implied by the following ones: 1) $S_N(M) \cap S_1M \neq \Lambda$; $\{\Lambda\}$ here $S_N(M) = S(M) \cap PN$; 2) there exists $a\psi \in P\varphi$ so that $pr_1\psi = SM - \{\Lambda, M\}$, $pr_2X \subseteq P_N(M)$, where

$$X = S_N(M) - \{\Lambda, M\}, \quad P_N M = P(M) \cap P(N).$$

One has then a "fundamental theorem" (second formulation) stating the necessary and sufficient conditions in order that SM be inductive for M relative to such a (S_1M, φ).

The paper contains numerous other definitions and statements partly connected with ordered sets. *G. Kurepa.*

Moneta, J. Récurrence transfinie de 1^re classe. Ann. Univ. Lyon. Sect. A. (3) 15, 17–25 (1952).

Cf. Eyraud, Cahiers Rhodaniens 5, 19–26 (1953); MR 15, 858. *F. Bagemihl* (Notre Dame, Ind.).

Kita, Tôru. A theorem on limit ordinals. Math. Japon. 3, 62 (1954).

The theorem in question is known [cf. Jacobsthal, Math. Ann. 66, 145–194 (1909), pp. 178, 181–182] and asserts that an ordinal λ is of the second kind if, and only if, $\lambda = \xi\lambda$ for any ξ satisfying $2 \leqq \xi < \omega$. *F. Bagemihl.*

Karl, Herbert. Das Wesen des Unendlichen in der Mathematik. Wiss. Z. Pädagog. Hochsch. Potsdam 1, 1–11 (1955).

Advocating the use of materialistic dialectic for the solution of problems connected with infinity, the author attempts to solve the continuum problem, but, through the use of a false analogy, is led to the absurd conclusion that the real numbers x satisfying $0 \leqq x \leqq 1$ constitute, in their natural order, a well-ordered set of type $\omega^\omega + 1$.
F. Bagemihl (Notre Dame, Ind.).

Reichbach, M. Une simple démonstration du théorème de Cantor-Bernstein. Colloq. Math. 3, 163 (1955).

If M is a set, and $f(M) \subset M$, where f is one-to-one, then, for every $E \subset M - f(M)$, there exists a function f^* that is one-to-one and such that $f^*(M) = E \cup f(M)$; indeed, if $S = E \cup f(E) \cup f[f(E)] \cup \cdots$, then f^* may be defined as follows: $f^*(x) = x$ for $x \in S$, $f^*(x) = f(x)$ for $x \in M - S$.
F. Bagemihl (Notre Dame, Ind.).

Havel, Václav. Remark on a generalization of the direct product of partially ordered sets. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 5, 3–10 (1955). (Czech. Russian summary)

Let A and B be partially ordered sets with order relations $\leqq$ and $\leqq_B$ respectively. Let $S = \bigcup_{b \in B} A_b$, where A_b is isomorphic to A under isomorphisms fixed for each $b \in B$. A relation r is defined in S as follows. For x, y in a single A_b, we have xry if and only if $x \leqq y$ (in $A_b \cong A$). For $x \in A_b$ and $y \in A_{b'}$, where $b <_B b'$, we have xry if and only if there exists $a \in A_b$ such that $x \leqq a$ and such that, under the isomorphism $A_b \cong A \cong A_{b'}$ of A_b onto $A_{b'}$, the relation $a' \leqq y$ holds, where a' is the image of a in $A_{b'}$. Conditions are found under which S is isomorphic to $A \times B$ and under which S is a lattice. *E. Hewitt* (Princeton, N. J.).

Urbanik, K. On plane sets composed of parallel segments. Prace Mat. 1, 169–173 (1955). (Polish. Russian and English summaries)

Let A be a measurable plane set whose vertical sections are single non-degenerate closed segments. (Sections are understood to be non-empty.) The author establishes a result suggested by H. Steinhaus: that it is possible to remove from A a set of verticals of measure zero, such that for the remaining set each horizontal section has positive linear measure. (The author's statement accidentally omits the hypothesis of measurability of A: this is a misprint.)
L. C. Young (Madison, Wis.).

Popović, V. Sur un théorème de N. Obrechkoff. Srpska Akad. Nauka. Zb. Rad. 43. Mat. Inst. 4, 57–61 (1955). (Serbo-Croatian. French summary)

The author proves that if $f^{(n)}(x) \to \delta_1$ $(x \to +\infty)$ and $f^{(n)}(x) \to \delta_2$ $(x \to -\infty)$, then $x^{-n}f(x) \to \delta_1/n!$ $(x \to +\infty)$ and $x^{-n}f(x) \to \delta_2/n!$ $(x \to -\infty)$. Using this, he gives a short proof and generalization of a theorem of Obrechkoff which deals with limit conditions forcing a given function to be a polynomial [Acta Sci. Math. Szeged 12, Pars B, 231–235 (1950); MR 11, 583]. *R. P. Boas, Jr.* (Evanston, Ill.).

Delange, Hubert. Sur un théorème de Karamata. Bull. Sci. Math. (2) 79, 9–12 (1955).

In a result of Karamata's [Mathematica, Cluj 4, 38–53 (1930)] a boundedness hypothesis may be dropped, yielding the following strengthened theorem. If the positive real-valued function L, defined for all real $x \geqq$ some x_0, is measurable on each closed interval $[x_0, x_1]$ and has

$$(*) \qquad \lim_{r \to \infty} \frac{L(rx)}{L(r)} = 1$$

for each positive x, then the convergence (*) is uniform in x on each interval $[a, b]$ with $0 < a < b$. *T. A. Botts.*

Hukuhara, Masuo. Sur la fonction convexe. Proc. Japan Acad. 30, 683–685 (1954).

Let f be a function convex in an open interval (α, β). A theorem of Ostrowski [Jber. Deutsch. Math. Verein. 38, Abt. 1, 54–62 (1929)] states that if f is bounded above on a subset of positive measure, then f is continuous in (α, β). On the other hand, it is easily seen that boundedness below, even in the whole interval, does not imply continuity. It is here proved that if f is bounded below on a subset of positive measure, then f is bounded below on (α, β).
F. F. Bonsall (Newcastle-on-Tyne).

Darbo, Gabriele. Convergenza in variazione e convergenza in lunghezza. Ann. Univ. Ferrara. Sez. VII. (N.S.) 3, 1–9 (1954).

A sequence $\{f_n(x)\}$ of real-valued functions defined on a closed interval $[a, b]$ is said to converge in variation (in length) to a function $f(x)$ on $[a, b]$ if $f_n(x)$ tends to $f(x)$ for each x in $[a, b]$ and $V_a{}^bf_n(x)$ tends to $V_a{}^bf(x)$ $(L_a{}^bf_n(x)$ tends to $L_a{}^bf(x))$. Adams and Lewy [Duke Math. J. 1, 19–26

(1935)] showed that convergence in length implies convergence in variation but not conversely. The present author derives a formula,

$$L_a{}^b f(x) = \frac{1}{2}\int_{-\infty}^{\infty} V_a{}^b(f(x)+\lambda x)(1+\lambda^2)^{-3/2}d\lambda,$$

related to Crofton's formula and uses it to show that for convergence in length of $f_n(x)$ to $f(x)$ it is necessary and sufficient that for all real λ, $f_n(x)+\lambda x$ converge in variation to $f(x)+\lambda x$. *M. M. Day* (Urbana, Ill.).

Maak, Wilhelm. Eine Verallgemeinerung des Weierstrassschen Approximationssatzes. Arch. Math. 6, 188–193 (1955).

The basic result is this. Let M be a set, f a real-valued function on M, $\varphi_1, \cdots, \varphi_n$ bounded functions on M, and assume there exist $\epsilon, \delta > 0$ such that $\max_i |\varphi_i(x)-\varphi_i(y)| < \delta$ implies $|f(x)-f(y)| < \epsilon$, x, y arbitrary in M. Then for $\eta > 0$ there is a polynomial $P(u_1, v_1; \cdots; u_n, v_n)$ such that $|f(x)-P(\varphi_1(x), \overline{\varphi_1(x)}; \cdots; \varphi_n(x), \overline{\varphi_n(x)})| < \epsilon+\eta$ for all x in M. The proof employs a simple construction. Let $F(u)=0$ outside $[0, \delta]$, $F(0)=1$, $F(\delta)=0$ and F linear in $[0, \delta]$. Let

$$H(u_1, \cdots, u_n; v_1, \cdots, v_n) = \prod_1^n F(|u_i - v_i|),$$

$$G(x, y) = H(\varphi_1(x), \cdots, \varphi_n(x); \varphi_1(y), \cdots, \varphi_n(y)).$$

Then a simple argument shows one can choose finitely many $a_j \in M$ such that

$$\sum_{j=1}^N G(x, a_j) \neq 0$$

and

$$\left| f(x) - \left(\sum_1^N f(a_i) G(x, a_i) \right) \Big/ \sum_1^N G(x, a_i) \right| \leq \epsilon$$

for all x. Using the classical (via Lebesgue) Weierstrass approximation theorem, the author gets the result directly. The form of this theorem leads to the proof of (a) the Stone-Weierstrass theorem for a compactum (modification of the author's argument is required for a general compact set), and (b) the uniform approximation theorems of almost periodic function theory. *B. Gelbaum.*

Calderón, Alberto P. Sur les mesures invariantes. C. R. Acad. Sci. Paris 240, 1960–1962 (1955).

Let μ be a countably additive measure in a space E. Let $G=\{g\}$ be a semi-group of transformations in E with the properties that $g^{-1}(A)$ is measurable if A is measurable and $\mu(g^{-1}(A))=0$ if $\mu(A)=0$. If there is a finitely additive function ν defined on the measurable sets and vanishing on precisely the μ-null sets and if ν is invariant under G, then there is a countably additive measure invariant with the same properties. This principle is applied to give a number of conditions which are sufficient for the existence of an invariant measure. For example, if G is generated by one element g, in order that there is an invariant measure, it is sufficient that one of the following conditions hold for every measurable set A:

$$\liminf_{n\to\infty} \mu[g^{-n}(A)] > 0; \quad \liminf_{n\to\infty} \sum_0^{n-1} \mu[g^{-k}(A)] > 0;$$

$$\liminf_{n\to\infty} \sum_{k=0}^{\infty} \mu[g^{-k}(A)] \left(\frac{n-1}{n}\right)^k > 0.$$

N. Dunford (New Haven, Conn.).

Bauer, Heinz. Darstellung additiver Funktionen auf Booleschen Algebren als Mengenfunktionen. Arch. Math. 6, 215–222 (1955).

Let $\mathfrak{B}'$ be an abstract Boolean algebra and $\mathfrak{B}$ a subalgebra of $\mathfrak{B}'$. Let f be an additive, non-negative function on $\mathfrak{B}$: $f(A \vee B) = f(A)+f(B)$ for all $A, B \in \mathfrak{B}'$ such that $A \wedge B = 0$, and $f(X) \geq 0$ for all $X \in \mathfrak{B}$. The function f is said to be σ-additive relative to $\mathfrak{B}'$ if for every sequence $\{A_i\}_{i=1}^{\infty}$ of pairwise disjoint elements of $\mathfrak{B}$ having a least upper bound A in $\mathfrak{B}'$ such that $A \in \mathfrak{B}$, the equality $f(A) = \sum_{i=1}^{\infty} f(A_i)$ obtains. The function f is said to be purely finitely additive relative to $\mathfrak{B}'$ if $0 \leq g \leq f$ and g σ-additive on $\mathfrak{B}$ relative to $\mathfrak{B}'$ imply that $g=0$. A simple construction is given to show that every additive f on $\mathfrak{B}$ can be written as $f_c + f_p$, where f_c is σ-additive relative to $\mathfrak{B}'$ and f_p is purely finitely additive relative to $\mathfrak{B}'$. This generalizes a theorem of K. Yosida and the reviewer [Trans. Amer. Math. Soc. 72, 46–66 (1952); MR 13, 543]. If $\mathfrak{B}$ is an algebra of subsets of a set E and $\mathfrak{B}'$ is the σ-algebra generated by $\mathfrak{B}$, then an additive function on $\mathfrak{B}$ that is σ-additive (purely finitely additive) relative to $\mathfrak{B}'$ is said to be set-theoretically σ-additive (purely finitely additive). An isomorphism Ψ of the abstract Boolean algebra $\mathfrak{B}'$ onto an algebra of subsets of a set E is said to be separating if for all $p, q \in E$, $p \neq q$, there exists $A \in \mathfrak{B}'$ such that $p \in \Psi(A)$ and q non-$\in \Psi(A)$.

The main theorems are the following. I. Let f, $\mathfrak{B}$, $\mathfrak{B}'$ be as above. Then f is σ-additive relative to $\mathfrak{B}'$ if and only if, under every separating isomorphic representation Ψ of $\mathfrak{B}'$ by an algebra of sets, the image $f\Psi^{-1}$ of f is set-theoretically σ-additive on $\Psi(\mathfrak{B})$. This theorem for the case $\mathfrak{B}=\mathfrak{B}'$ was proved by the reviewer [Duke Math. J. 20, 253–256 (1953); MR 14, 854]. II. The function f is purely finitely additive relative to $\mathfrak{B}'$ if and only if there exists a separating isomorphism Ψ of $\mathfrak{B}'$ onto an algebra of sets such that the function $f\Psi^{-1}$ is set-theoretically purely finitely additive on the algebra $\Psi(\mathfrak{B})$. *E. Hewitt* (Princeton, N. J.).

Bledsoe, W. W., and Morse, A. P. Product measures. Trans. Amer. Math. Soc. 79, 173–215 (1955).

S: groundset. For $A \subset S$, c_A: characteristic function of A. ϕ, ψ: Carathéodory (outer) measures on S. If $\mathfrak{C}$ denotes any family of subsets of S, $\phi_{\mathfrak{C}}(A) = \inf_{H \in \mathfrak{H}} \sum \phi(H)$ for all countable subfamilies $\mathfrak{H}$ of $\mathfrak{C}$ covering A, the infimum $= +\infty$ if there does not exist such a countable covering family. §1 is the introduction. §2 contains preliminary definitions and notations. In §3 is considered a $s\delta$-family $\mathfrak{F}$ of subsets of S, i.e., a family closed with respect to finite union and countable nonvacuous intersection, covering S (a "pseudotopology" in Nikodym's terminology), and the family $\mathfrak{G}$ of the S-complements of the $\mathfrak{F}$-sets. ϕ is $\mathfrak{F}$-adapted if (i) the $\mathfrak{F}$-sets are measurable, (ii) corresponding to any set $G \in \mathfrak{G}$, any $T \subset S$ with finite $\phi(T)$, and any $\epsilon > 0$, there exists in $\mathfrak{F}$ a subset C of G such that $\phi(TG - TC) < \epsilon$. (ii) follows from (i) if $\mathfrak{G}$ is included in the Borel extension of $\mathfrak{F}$ (smallest $\delta\sigma$-family over $\mathfrak{F}$). If B is $\mathfrak{F}$-universally measurable, i.e., measurable with respect to each $\mathfrak{F}$-adapted (outer) measure, and if ϕ is such a measure for which $\phi(B)$ is finite, B can be approximated from inside by $\mathfrak{F}$-sets. The family of the $\mathfrak{F}$-universally measurable sets includes the family of the analytic sets over $\mathfrak{F}$. No example is given showing if it could be strictly more extensive.

In §4 ϕ is first a general (outer) measure on S. A (numerical) function is ϕ-integrable if it vanishes outside of a σ-finite subset of S, is ϕ-measurable and the integrals of the

positive and negative parts are not both infinite. A $s\delta$-family $\mathfrak{F}$ of subsets of S is introduced. The sets of $\mathfrak{F}$ are assumed to be ϕ-measurable and σ-finite, $\phi=\phi_{\mathfrak{F}}$. An $\mathfrak{F}$-weight function h is defined as follows: Let $\mathfrak{H}$ be any countable subfamily of $\mathfrak{F}$, $\lambda(H)$ a positive (possibly infinite) number attached to each $H\in\mathfrak{H}$; then $h(x)=\sum_{H\in\mathfrak{H}}\lambda(H)\cdot c_H(x)$ in each point x of S. Fundamental representation theorem: Corresponding to any ϕ-integrable function f, there exist two $\mathfrak{F}$-weight functions h' and h'' such that $f(x)=h'(x)-h''(x)$ ϕ-almost everywhere on S and

$$\int f(x)\phi(dx)=\int h'(x)\phi(dx)-\int h''(x)\phi(dx).$$

In §5 μ denotes an (outer) measure on a set X, ν an outer measure on a set Y, $S=X\times Y$. $\mathfrak{F}$ denotes an $s\delta$-family of subsets of S covering S such that the characteristic function of each set F of $\mathfrak{F}$ integrates iteratively in both orders to the same number denoted by $(\mu\nu)_{\mathfrak{F}}(F)$, briefly $\psi(F)$. For any subset A of S, $(\mu\nu)_{\mathfrak{F}}(A)=\inf_{F\in\mathfrak{H}}\sum\psi(F)$ for all countable subfamilies $\mathfrak{H}$ of $\mathfrak{F}$ covering A. Finally the sets of $\mathfrak{F}$ are assumed to be $(\mu\nu)_{\mathfrak{F}}$-measurable and σ-finite. Fubini's theorem is established for $(\mu\nu)_{\mathfrak{F}}$-integrable functions using the representation theorem of §4. An interval I is the product $M\times N$ where M is a μ-measurable set with $\mu(M)<\infty$ and N a ν-measurable set with $\nu(N)<\infty$, $g(I)=\mu(M)\cdot\nu(N)$. The key of the (pseudo-)topology-free measure product $\phi=(\mu\nu)$ is the following notion: a nilset is a subset of S whose characteristic function integrates iteratively in both orders to zero. $\phi(A)$ is defined as the infimum of the numbers $\sum_{I\in\mathfrak{J}}g(I)$ where $\mathfrak{J}$ is any countable family of intervals covering A but for a nilset. The product of a μ-measurable set with a ν-measurable set is proved to be $(\mu\nu)$-measurable, the nilsets are $(\mu\nu)$-nullsets and Fubini's theorem holds.

In §6 $\mathfrak{T}$ denotes a topology on S, precisely the family of the open sets, $\mathfrak{F}$ the family of the closed sets. Several types of topological measures are considered. The most special one is the Radon measure on a locally compact regular topological space. A general topological measure is an $\mathfrak{F}$-adapted measure ψ on a general topological space such that corresponding to any covering of S by open sets and any $T\subset S$ with finite $\psi(T)$, there exists a countable subfamily covering T but for a ψ-nullset. $\mathfrak{E}$ denoting a σ-family containing with each set its relatively open subsets, the properties of $\phi_{\mathfrak{E}}$ when ϕ is a topological measure, are investigated, especially when $\mathfrak{E}=\mathfrak{T}$. Locally nullsets are examined briefly. Some attention is paid to metric measures. In §7 μ and ν are two general topological measures in the sense of §6 defined on the sets X and Y respectively. It is proved that the topology-free product measure $\phi=(\mu\nu)$ defined in §5 is a topological measure in the topological product space. The usual difficulty met with in proving adaptation theorems of that kind is to define the product measure in such a way as to make any open set ϕ-measurable. Saks' definition of a product measure in his "Theory of the integral" [2nd ed., Warszawa-Lwów, 1937, p. 85–87] restricts a priori the family of measurable sets in the product space; thus in the present case the family may not contain all open sets of $X\times Y$ if X and Y do not each admit a countable topological basis. Halmos' definition yields the desired result for two Radon measures. The novel feature of the present theory consists in extending Saks' product measure by means of the nilsets under an adequate Lindelöf property. The authors' definition gives to the Fubini theorem a maximal range of validity. *C. Y. Pauc.*

Takahashi, Shigeru. Notes on the Riemann-sum. Proc. Japan Acad. **31**, 8–13 (1955).

Let t_i be independently and uniformly distributed and let f be integrable or square-integrable in $(0, 1)$ and of period 1. Under certain local order conditions it is shown that the Riemann sums of f formed with the t_i's arranged in increasing order converges to its integral over $(0, 1)$. Since f is supposed to be Borel-measurable it seems that the two conclusions of the two theorems are equivalent by Fubini's theorem. *K. L. Chung* (Syracuse, N. Y.).

McShane, E. J. A dominated-convergence theorem. Duke Math. J. **22**, 325–331 (1955).

In his book, "Order-preserving maps and integration processes" [Princeton, 1953 (quoted [2]); MR **15**, 19] the author generalized Daniell's integral theory, giving the essential role to the order properties. He considers namely an elementary integral as an order-preserving (isotone) mapping I_0 of a subset E of some lattice F into a partially ordered set G and extends it to a mapping I of $F_0\supset E$ into G in such a way that I enjoys the lattice and order-convergence properties of the classical Lebesgue integral. Typical in this regard is the Lebesgue dominated-convergence theorem. However in [2] properties extraneous to an order structure are assumed: G is an additive group and "normal." The purpose of the present note is to establish the lattice property of the set of summable elements and a generalized dominated-convergence theorem under hypotheses weaker than in [2] and expressible in terms of order alone. The setting of chapters I and II of [2] is taken over. A syntax is the image of a filter-basis of a mapping ("Nöbeling's gerasterte Funktion"). For any two subsets A and B of F, $A[\vee]B$ means the set $\{a\vee b: a\in A \text{ and } b\in B\}$. If $\mathfrak{A}$ and $\mathfrak{B}$ are families of subsets of F, $\mathfrak{A}[\vee]\mathfrak{B}$ means the collection of sets $\{A[\vee]B: A\in\mathfrak{A} \text{ and } B\in\mathfrak{B}\}$. To secure the summability of $u_1\vee u_2$ for any two summable elements u_1 and u_2, the following property is postulated: Let l^* be a summable L-element, and let $\mathfrak{A}$, $\mathfrak{B}$ be filter-bases of sets of L-elements all $\geqq l^*$. Let each $A\in\mathfrak{A}$ and each $B\in\mathfrak{B}$ be directed both by $\geqq$ and by $\leqq$. Let the syntaxes $(I_1, \mathfrak{A})$ and $(I_1, \mathfrak{B})$ be convergent to points of G. Then $(I_1, \mathfrak{A}[\vee]\mathfrak{B})$ is also convergent to a point of G. The dual also holds for filter-bases of sets of U-elements all $\leqq$ a fixed summable U-element. In order to establish the preliminary monotone-convergence theorem, a weakened form of the Hausdorff separation property is assumed, namely: For each pair a, b of distinct points of G such that $a\leqq b$, there exist disjoint sets Va, Vb containing a and b respectively and open with respect to the author's order-topology. Under all postulates listed we have: Let f', f'', f_1, f_2, $\cdots$ be summable elements such that $f'\leqq f_n\leqq f''$, $n=1, 2, 3, \cdots$. If $\lim_{n\to\infty} f_n$ exists, it is summable, and $I(\lim_{n\to\infty} f_n)=\lim_{n\to\infty} I(f_n)$. *C. Y. Pauc* (Nantes).

Phakadze, Š. S. On iterated integrals. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **20**, 167–209 (1954). (Russian)

This paper deals with the dependence of iterated Lebesgue integrals on the order of integration. By decomposing the range of the inner variable of integration into a finite set of subintervals and summing outside the outer integral the author defines, for bounded functions on a rectangle R, "strong" iterated upper and lower integrals. He then proves that when the strong iterated upper and lower xy (or yx) integrals are applied to the characteristic functions of subsets of R they define related normal Carathéodory outer and inner measures. A set M is said to have property A if

the xy and yx outer and inner measures all agree on M. It is proved that if almost every section of M parallel to the axes is open or closed, then M has property A, but that, under the hypothesis of the continuum, there is a set M whose sections parallel to one axis are of Baire class 0 and sections parallel to the other are of class 1, while the xy and yx integrals of the characteristic function of M are unequal.
M. M. Day (Urbana, Ill.).

Theory of Functions of Complex Variables

Bajšanski, Bogdan. Sur les zéros de la dérivée d'une fonction rationnelle. Srpska Akad. Nauka. Zb. Rad. **43**, Mat. Inst. **4**, 131–134 (1955). (Serbo-Croatian. French summary)

Let $f(z)$ be a rational function, P the smallest convex polygon containing all of its zeros, and Q the [smallest convex polygon containing all of its poles. If $P\cap Q$ is empty then all of the zeros of $f'(z)$ lie in $R\cup S$ where R is P plus its shadow when the points of Q are considered as sources of light, and S is defined symmetrically. This result is contained as a special case in Bôcher's Theorem [J. L. Walsh, The location of critical points of analytic and harmonic functions, Amer. Math. Soc. Colloq. Publ., v. 34, New York, 1950, p. 97; MR **12**, 249].
A. W. Goodman.

Tomić, M. Remarque sur les zéros d'une classe des fonctions méromorphes. Srpska Akad. Nauka. Zb. Rad. **43**, Mat. Inst. **4**, 73–80 (1955). (Serbo-Croatian. French summary)

The author proves three theorems on the zeros of certain meromorphic functions. A typical one is: if

$$f(z)=\sum_{-\infty}^{+\infty}(A_n(z-a_n)^{-1}+\lambda_n),$$

where $A_n\geq 0$, $\lambda_n\geq 0$, and all of the poles lie on a line L, not parallel to the real axis, then $f(z)$ has no zeros to the right of L. A number of applications are made.
A. W. Goodman (Lexington, Ky.).

On order

***Lekkerkerker, C. G. On the zeros of a class of Dirichlet series.** Van Gorcum & Comp. N.V., Assen, 1955. v+65 pp. 6.90 florins.

This dissertation extends known facts concerning the distribution of the zeros of the Riemann zeta faction to a class of Dirichlet series which includes the (λ,κ,γ)-functions of Hecke. A class of functions $\varphi(s)$ has the property P if (1) there exists an allied function $\varphi^*(s)$ such that $\varphi(s)$ and $\varphi^*(s)$ are represented by convergent Dirichlet series $\sum a_n n^{-s}$ and $\sum a_n^* n^{-s}$ in some right half-plane and are holomorphic outside of some rectangle; (2) there exists real β, h and a positive μ such that $R(s)=R^*(2h-s)$ outside the rectangle where $R(s)=e^{-\beta s}\Gamma(\mu s)\varphi(s)$, $R^*(s)=e^{-\beta s}\Gamma(\mu s)\varphi^*(s)$; and (3) $\log|\varphi(s)|=O(|s|^K)$ for all large $|s|$ and some fixed $K>0$. Such a function $\varphi(s)$ has trivial negative real zeros, the distance between consecutive zeros being $1/\mu$, all other zeros lie in some vertical strip and the number of zeros in the strip $0<t<T$ (and in $-T<t<0$) is

$$N(T)=\frac{1}{\pi}\mu T\log T-\frac{1}{\pi}[\tfrac{1}{2}\log cd+\beta+\mu-\mu\log\mu]T+O(\log T),$$

where c and d are the subscripts of the first non-vanishing coefficients a_n and a_n^* respectively. For a (λ,κ,γ)-function $\mu=1$ and the bracket equals $1+\log(2\pi c/\lambda)$. If $\varphi(s)$ is an entire $(1,\kappa,\gamma)$-function and $\chi=\chi(n)$ is a character modulo q, similar formulas hold for the functions $\varphi(s,\chi)=\sum a_n\chi(n)n^{-s}$. If now (i) $\varphi(s)$ is holomorphic and of finite order for $\sigma\geq h$, (ii) $R(s)\equiv ae^{-bs}\Gamma(s)\varphi(s)$ is real on $\sigma=h$, and

(iii) $$|f(x+iy)|=O(x^{-k})$$

as $x\to 0+$ where $f(z)=\sum a_n e^{-nz}$, then the number of zeros $N_0(T)$ on $\sigma=h$, $0<t<T$ (or $-T<t<0$), exceeds a constant multiple of T. It is shown that if $\varphi(s)$ satisfies (i) then it can satisfy (ii) if and only if property P holds with $\mu=1$ and $a_n^*=\gamma\bar{a}_n$. In particular, $N_0(T)>AT$ for every entire (λ,κ,γ)-function with real coefficients and $0<\lambda<2$. Here $h=\frac{1}{2}\kappa$.
E. Hille (New Haven, Conn.).

Have

***Korevaar, Jacob. Entire functions as limits of polynomials.** Lectures on functions of a complex variable, pp. 421–423. The University of Michigan Press, Ann Arbor, 1955. $10.00.

Summary of results in Duke Math. J. **21**, 533–548 (1954); MR **16**, 347.

Baker, I. N. The iteration of entire transcendental functions and the solution of the functional equation $f\{f(z)\}=F(z)$. Math. Ann. **129**, 174–180 (1955).

The author shows first that (*) $f\{f(z)\}=F(z)$ has no entire solution f if $F(z)$ is an entire function of finite order, bounded on a curve extending to ∞. From this he deduces conditions which ensure that an analytic solution of (*) cannot be single-valued, and in particular that Kneser's solution [J. Reine Angew. Math. **187**, 56–67 (1949); MR **11**, 726] of $f\{f(z)\}=e^z$, which is analytic on the whole real axis, is not single-valued. To illustrate what is possible, the author constructs transcendental entire functions f_1, f_2, f_3, of zero order, such that $f_1\{f_1\}$ is of zero order, $f_2\{f_2\}$ is of infinite order, $f_3\{f_3\}$ is of finite nonzero order.
R. P. Boas, Jr. (Evanston, Ill.).

Jung, Hans Peter. Beiträge zur Theorie der schlichten Funktionen. Mitt. Math. Sem. Giessen no. **52**, ii+29 pp. (1955).

Let $f(z)=z+a_2z^2+\cdots+a_nz^n$ be a normalized polynomial of degree n, schlicht in $|z|<1$. Using a theorem of G. Szegö [Math. Z. **13**, 28–55 (1922)], the author obtains inequalities for the last three coefficients $|a_n|\leq 1/n$, $|a_{n-1}|\leq 1$, $|a_{n-2}|\leq\frac{1}{2}(n-1)$, and with the aid of the criterion of Schur obtains also $|a_{n-1}|\leq 6(n-1)^{-1}$.

Let (S) denote the class of functions

$$f(z)=z+a_2z^2+\cdots+a_nz^n+\cdots$$

regular and schlicht in $|z|<1$. Denote $A_n=\max_{f\subset(S)}|a_n|$. The author gives another proof for the coefficient relation

$$(n+1)a_{n+1}-2a_2a_n=(n-1)\bar{a}_{n-1}e^{2i\arg a_n},$$

obtained by F. Marty [C. R. Acad. Sci. Paris **198**, 1569–1571 (1934)] for the coefficients of the corresponding extremal function of (S), and deduces

$$|a_{n-1}|\leq|a_n|,\quad |a_{n+1}|\leq(n+3)(n+1)^{-1}|a_n|$$

for the extremal function.

The third section of the paper discusses integral representations for the coefficients a_n of $f(z)$ in (S) obtained by means of the Schwarz-Christoffel mapping formula which are compared with those of the Löwner type. Bounds for $|a_2|$ and $|a_3|$ are obtained in special cases. Another proof for $|a_3|\leq 3$ is given, making use of recent results of O.

Tammi [Ann. Acad. Sci. Fenn. Ser. A. nos. **149**, **162** (1953); MR **15**, 302, 516]. *M. S. Robertson.*

Uluçay, Cengiz. Sur les fonctions de Bloch de la troisième espèce. Comm. Fac. Sci. Univ. Ankara. Sér. A. **6**, 5–10 (1954). (Turkish summary)

Uluçay, Cengiz. Sur les fonctions de Bloch de la première et de la seconde espèce. Comm. Fac. Sci. Univ. Ankara. Sér. A. **6**, 11–16 (1954). (Turkish summary)

Uluçay, C. On the constant $\mathfrak{C}$. Comm. Fac. Sci. Univ. Ankara. Sér. A. **6**, 77–88 (1954). (Turkish summary)

R. M. Robinson [Duke Math. J. **2**, 453–459 (1936)] defined Bloch functions (B.F.) as follows: Let C be the upper bound of constants P such that every admissible function f assumes the interior of a circle of radius P properly. The B.F. are the extremals for which the value C is assumed. For B.F. of the first and second kinds, "admissible" means holomorphic in $|z|<1$ with the normalization $f(z)=z+\cdots$, while for B.F. of the third kind, univalency is also required. For B.F. of the second and third kinds, "properly assumes" means "assumes", whereas for B.F. of the first kind, it means "assumes univalently". Robinson stated (and proved only special cases of) the following theorems. Every B.F. of the third kind maps $|z|<1$ conformally onto a domain R such that every point of the plane is either interior to or on the boundary of R. All B.F. of the first and second kinds map $|z|<1$ onto an open Riemann surface without boundary. In the first two papers, the author proves completely these two theorems. For B.F. of the first kind, $C=\mathfrak{B}$; second kind, $C=\mathfrak{L}$; and third kind, $C=\mathfrak{A}$. The author introduces a new constant $\mathfrak{C}$ such that $\mathfrak{L}<\mathfrak{C}\leqq\mathfrak{A}$. In the third paper, he shows that $\mathfrak{C}\leqq 0.64$. *G. Springer* (Lawrence, Kan.).

Springer, George. Extreme Punkte der konvexen Hülle schlichter Funktionen. Math. Ann. **129**, 230–232 (1955).

The functions $f(z)=a_{-1}z+\sum_1^\infty a_nz^{-n}$, $|z|>1$, form a linear space F which may be normed by $\|f\|=\{\sum_1^\infty n|a_n|^2\}^{1/2}$ (assumed to be finite). S is the subset of all schlicht f with $a_{-1}=1$, and K_s is its convex hull. If $I\equiv z$, then K_s lies in the convex intersection Σ of $\{a_{-1}=1\}$ and $\|f-I\|\leqq 1$, by the area-theorem. It is shown that an $f\in S$ is extremal for K_s if the complement of the map of $|z|>1$ by f has area zero, and that f is not extremal if this map is not everywhere dense [both results follow immediately from the area theorem]. It is also shown that Σ is not identical with the closure of K_s. *W. W. Rogosinski.*

Evgrafov, M. A. The Abel-Gončarov interpolation problem for nodes disposed upon a given ray. Dokl. Akad. Nauk SSSR (N.S.) **101**, 789–791 (1955). (Russian)

The author generalizes the Abel interpolation series by replacing $F^{(n)}(n)$ by $F^{(n)}(\lambda(n))$, where $\lambda(n)$ is twice differentiable and $n\lambda'(n)/\lambda(n)\to 1/\rho$. By using generalized Laplace-Borel transforms associated with $\lambda(n)$, he obtains sufficient conditions for the representation of a given $F(z)$; for the special case $\lambda(n)=n^{1/\rho}$ he obtains a necessary condition and also a uniqueness theorem. *R. P. Boas, Jr.*

Evans, J. P., and Walsh, J. L. On interpolation to a given analytic function by analytic functions of minimum norm. Trans. Amer. Math. Soc. **79**, 158–172 (1955).

Let the region R_1 of the z-plane contain the points $\beta_{n1}, \beta_{n2}, \cdots, \beta_{nn}$, where $n=1, 2, \cdots$, let $f(z)$ be analytic in them, and let $g_n(z)$ be such that, among all the functions which are analytic throughout R_1 and are equal to $f(z)$ at each β_{nk} (n fixed; $k=1, 2, \cdots, n$), it has the least norm in R_1. The convergence of $g_n(z)$ to $f(z)$, as $n\to\infty$, is studied. The case when the norm is defined as $\sup|g_n(z)|$ ($z\in R_1$) was discussed in a previous paper [Walsh, same Trans. **46**, 46–65 (1939); MR **1**, 10]. Here the norm N is measured by (1) a surface integral, $N=(\int\int_{R_1}|g_n(z)|^q dS)^{1/q}$ $(1<q<\infty)$, or (2) a parametric integral over the boundary C_1 of R_1, $N=\int_{C_1}|g_n(z)|^q d\psi(z))^{1/q}$ $(0<q\leqq\infty)$, or (3) a line integral over C_1. Both in the previous and in this paper, the condition is added that the β_{nk} should lie in some closed point set $\bar{R}_0$ in R_1 and that $\lim|(z-\beta_{n1})\cdots(z-\beta_{nn})|^{1/n}=e^{V_1(z)}$ should exist uniformly on any closed bounded set in the complement of $\bar{R}_0$. Concerning (3), the function $H^*_{n1}(z)$ is found which is of minimum norm $(\int_{C_1}|H^*_{n1}(z)|^q|dz|)^{1/q}$ $(1<q<\infty)$ among all the $H_{n1}(z)$ for which

$$H_n(z)=(2\pi i)^{-1}\int_{C_1}H_{n1}(w)(w-z)^{-1}dw$$

is equal to $f(z)$ at the points β_{nk} $(k=1, \cdots, n)$, and then the convergence of $H_n^*(z)$ to $f(z)$, as $n\to\infty$, is deduced, where $H_n^*(z)=(2\pi i)^{-1}\int_{C_1}H^*_{n1}(w)(w-z)^{-1}dw$.

The results are analogous to those of the previous paper. If now $q=2$ and the points β_{nk} reduce to $\beta_1, \beta_2, \cdots, \beta_n$, the general theory of orthogonal functions can be used: an expansion of $f_n(z)$ in a series of orthonormal $\phi_n^*(z)$ is obtained and its convergence properties are discussed in detail. A result proved by Walsh and Davis, but in a different way [J. Analyse Math. **2**, 1–28 (1952); MR **16**, 580], for the special case when $\lim\beta_n$ exists, is generalised (see Theorem 4.1). Finally the asymptotic behavior of the $\phi_n^*(z)$ and of their zeros is investigated. *H. Kober* (Birmingham).

Heins, Maurice. On the Lindelöf principle. Ann. of Math. (2) **61**, 440–473 (1955).

Mémoire fondamental concernant les représentations conformes f d'une surface de Riemann F dans une autre G. Soit $p\in F$, $n(p)$ l'ordre de multiplicité de f en p. Soit $q\in G$ et $\mathfrak{G}_G$ et $\mathfrak{G}_F$ des fonctions de Green de G et F respectivement. On peut écrire

$$\mathfrak{G}_G(f(p);q)=\sum_{f(r)=q}n(r)\mathfrak{G}_F(p;r)+u_q(r).$$

P désignant l'ensemble des fonctions harmoniques (f.h.) $\geqq 0$ sur F on a $u_q(r)\in P$; en outre $u_q(p)=\mathrm{GHM}\ \mathfrak{G}_G(f(p);q)$ (où GHM = plus grande minorante harmonique). Après ces préliminaires l'auteur déduit d'intéressants corollaires d'un théorème de Kjellberg sur les fonctions $\in P$.

Soit alors $\Omega\subset F$ et P_Ω la classe des f.h. $\geqq 0$ sur Ω et nulles sur fr Ω; $u\in P\to\lambda_\Omega(u)\in P_\Omega$ de la manière suivante: $\lambda_\Omega(u)$ est l'enveloppe supérieure des $U\in P_\Omega$ et vérifiant $U\leqq u$ (sur Ω). L'auteur étudie de très remarquables propriétés de λ_Ω et de son inverse μ_Ω d'ailleurs défini autrement.

La troisième partie est consacrée à l'étude de $u_q(p)=v_q+w_q$ où $v_q\in P$, $w_q\in P$ avec v_q quasi bornée et w_q singulière (la seule fonction de P bornée et $<w_q$ est 0). Sur $F\times G$ ou bien $v_q(p)>0$ ou bien $v_q(p)=0$ d'où l'on déduit que ou bien pour tout $q\in G$, $u_q\geqq$ une f.h. >0, ou bien cela n'a lieu pour aucune valeur de q. Ce dernier cas fait l'objet d'une étude très complète dans la suite de cette partie; les applications f possédant cette propriété sont dites de type Bl (lorsque F et G sont deux disques-unité, f est le produit d'une fonction homographique par un produit de Blaschke).

Dans la quatrième partie Φ désigne la famille des domaines de Jordan simplement connexes et contenant $q\in G$. Soit $\Omega\in\Phi$. $\sigma(\Omega)$ est une composante de $f^{-1}(\Omega)$ telle que si $\Omega_1\subset\Omega_2$, $\sigma(\Omega_1)\subset\sigma(\Omega_2)$. Un filtrage asymptotique en q ("asymptotic

spot" que nous noterons A.S.) est une fonction σ telle que $\sigma(\Omega)$ ne soit relativement compacte (sur F) pour aucun $\Omega \in \Phi$. q est alors valeur asymptotique pour f. Soient alors F et G deux surfaces de Riemann à frontières positives. Si $\Omega \in \Phi$ soit $U_{\sigma,\Omega} = \mu_{\sigma(\Omega)}[\text{GHM } \mathfrak{G}_0(f_{\sigma(\Omega)}; q)]$ où $f_{\sigma(\Omega)}$ est la restriction de f à $\sigma(\Omega)$. Soit U_σ = enveloppe inf $U_{\sigma,\Omega}$ (lorsque Ω parcourt Φ). U_σ est harmonique $\geqq 0$ et singulière. Un A.S. est dit métrique (M.A.S.) si $U_\sigma > 0$. L'ensemble des M.A.S. en q est dénombrable. Dans certains cas particulièrement importants et pour des M.A.S. la fonction U_σ est minimale (c'est à dire $v \in P$ et $v < U_\sigma$ entraînent $v = kU_\sigma$): Ce cas se présente par exemple si F est de genre fini. L'auteur démontre aussi que l'ensemble des valeurs asymptotiques métriques est un F_σ de capacité nulle.

Les résultats précédents trouvent de remarquables applications dans l'étude des f localement de type Bl (certaines propriétés démontrées à la fin de la troisième partie permettent de définir cette nouvelle notion). La composition d'une f.h. >0 singulière avec une application localement Bl est une f.h. >0 singulière. L'auteur obtient aussi une caractérisation des applications localement Bl par la capacité nulle de l'ensemble des valeurs asymptotiques.

L. Fourès (Princeton, N. J.).

Tietz, Horst. Eine Normalform berandeter Riemannscher Flächen. Math. Ann. **129**, 44–49 (1955).

The following theorem is proved: Any open Riemann surface of finite genus which has n non-degenerate boundary components can be mapped conformally onto a Riemann surface consisting of a number of full planes and n unit disks.

Z. Nehari (Pittsburgh, Pa.).

***Ohtsuka, Makoto. Boundary components of abstract Riemann surfaces.** Lectures on functions of a complex variable, pp. 303–307. The University of Michigan Press, Ann Arbor, 1955. $10.00.

Summary of results in Nagoya Math. J. **7**, 65–83 (1954); MR **16**, 349.

Bader, Roger. Fonctions à singularités polaires sur des domaines compacts et des surfaces de Riemann ouvertes. Ann. Sci. Ecole Norm. Sup. (3) **71**, 243–300 (1954).

The existing literature on differentials on an abstract Riemann surface has been confined to those with at most a finite number of poles. The author is the first to attack the case where infinitely many poles are present. He tells us his approach is inspired by Ahlfors' method [Comment. Math. Helv. **24**, 100–134 (1950); MR **12**, 90; **13**, 1138].

Given an open Riemann surface S, consider a nested sequence of exhausting compact subregions G with analytic boundaries β. The author uses the term Schottky-Ahlfors differential for a meromorphic differential on G which permits a suitable continuation to the double, $\hat{G}$, of G with respect to β. He shows that every meromorphic differential Ω on G can be decomposed into a sum of an exact differential of the first kind and a Schottky-Ahlfors differential ω. If the square root of the Dirichlet integral is taken as the norm, then this decomposition can be so chosen that ω differs arbitrarily little from Ω. Given a meromorphic differential d on S, it is possible to select its Schottky-Ahlfors components d_G on the G such that d appears, up to an analytic differential of finite norm, as the uniform limit of the d_G.

Denote by C_a the class of meromorphic differentials Ω on S which are the limits of their suitably chosen Schottky-Ahlfors components on the G. For every Ω in C_a there exists a set $D \subset S$ with compact components which contains the poles of Ω and has the property $\|\Omega\|^2_{S-D} + \|\Omega-\omega\|_D^2 < \infty$, where ω is a component of Ω in D. If S belongs to the class O_{AD} of Riemann surfaces without nonconstant analytic functions of finite norm, then the differentials Ω are uniquely determined by their singularities and periods.

Similar results are obtained for closed real differentials with polar singularities. Every such differential on G permits a decomposition into a sum of a harmonic differential with polar singularities (the Schottky-Ahlfors component) and an exact differential. Consequently, any harmonic differential on S can be represented as the sum of an exact harmonic differential with finite norm and the limit of suitably chosen Schottky-Ahlfors components on the G. The harmonic differentials constituting the analogue C_h of the class C_a are uniquely determined by their singularities and periods if S is in the class O_{HD} of Riemann surfaces without harmonic functions of finite norm. A sufficient condition is obtained for the existence on a parabolic surface of an exact differential in C_h with prescribed singularities.

Meromorphic multiplicative functions with multipliers with unit modulus are then discussed. In particular, such functions with unit modulus (class M_1) or constant argument on the boundary are considered on G. A multiplicative function can be expressed as the product of a function in M_1 and a function without zeros or poles on G. The bases of differentials of the first kind on the $\hat{G}$ converge to a base of such differentials on S. This result permits an extension of Abel's theorem to suitably restricted multiplicative functions and integrals of harmonic differentials on parabolic Riemann surfaces.

L. Sario (Los Angeles, Calif.).

Endl, Kurt. Zum Typenproblem Riemannscher Flächen. Mitt. Math. Sem. Giessen no. **49**, i+35 pp. (1954).

The author considers simply connected Riemann surfaces W with branch points over a finite number q of points a_k of the extended complex plane. Let M_k be an upper bound for the multiplicities of branch points covering a_k. The author shows that the condition

$$(1) \qquad \sum_{k=1}^{q} (1-(1/M_k)) = 2$$

is sufficient for W to be of parabolic type. The same is true if the branch points are shifted so as to lie above disjoint disks D_k centered at a_k; then M_k appears as an upper bound for the numbers m_{ki} of sheets in regions covering D_k.

For the proof, W is represented by a line complex, and the number $\sigma_0(n)$ of free boundary knots is shown to be maximized, under condition (1), by surfaces whose numbers of sheets are precisely M_k. To these extremal surfaces the Wittich criterion $\sum(1/\sigma_0(n)) = \infty$ [Math. Z. **45**, 642–668 (1939); MR **1**, 211] is shown to apply, and the parabolicity of W follows.

Condition (1) is best possible, for the Ahlfors test [Acta Math. **65**, 157–194 (1935)] guarantees hyperbolic type if the sum (1) for corresponding lower bounds M_k exceeds 2.

L. Sario (Los Angeles, Calif.).

Kuramochi, Zenjiro. Dirichlet problem on Riemann surfaces. I. Correspondence of boundaries. Proc. Japan Acad. **30**, 731–735 (1954).

A Riemann surface R is given as a covering, $z^* = f(z)$, of another Riemann surface R^*. Suppose the universal covering surface R^∞ of R is conformally mapped by the function $\xi = \varphi(z)$ onto the disk $|\xi| < 1$. If the covering $f(z)$ possesses suitable boundedness properties, then it is shown that the

function $z^*=f(\varphi^{-1}(\xi))$ has angular limits almost everywhere on $|\xi|=1$, and that there exists a set E of measure 2π on $|\xi|=1$ such that every Stolz path terminating at E determines, in a specified sense, an accessible boundary point of R.

L. Sario (Los Angeles, Calif.).

Kuramochi, Zenjiro. Dirichlet problem on Riemann surfaces. II. Harmonic measures of the set of accessible boundary points. Proc. Japan Acad. **30**, 825–830 (1954).

Consider the case where R is hyperbolic and R^* parabolic. Let $A(R, \bar{R}^*)$ be the set of accessible boundary points of R with respect to the union $\bar{R}^*$ of R^* and its ideal boundary components. Denote by $\mu(R, A(R, \bar{R}^*))$ the lower envelope of all continuous super-harmonic functions $v(z)$ with $0\leq v(z)\leq 1$ on R, such that $\lim v(z)=1$ as z tends to $A(R, \bar{R}^*)$. The analogues of A and μ are defined for the universal covering surface R^∞ of R. The arcs on the unit circle of the normal polygon under the uniformization of R are designated by α_i, the substitutions in the Fuchsian group, by T_j.

The author shows that, if the measure of $\sum_j (T_j(\sum_i \alpha_i))$ is 2π, then $\mu(R^\infty, A(R^\infty, \bar{R}^*))=\mu(R, A(R, \bar{R}^*))$. A similar result is derived for closed subsets of $A(R, \bar{R}^*)$.

L. Sario (Los Angeles, Calif.).

Kuramochi, Zenjiro. Dirichlet problem on Riemann surfaces. III. Types of covering surfaces. Proc. Japan Acad. **30**, 831–836 (1954).

The author defines various kinds of coverings R of R^* in terms of μ and $f(z)$. With respect to these types he then discusses relations between R and its coverings $\hat{R}$.

L. Sario (Los Angeles, Calif.).

Kuramochi, Zenjiro. Dirichlet problem on Riemann surfaces. IV. Covering surfaces of finite number of sheets. Proc. Japan Acad. **30**, 946–950 (1954).

If R consists of a finite number of sheets and R^* is parabolic, then it is shown that all accessible boundary points are regular for the Dirichlet problem save for a subset of $A(R, \bar{R}^*)$ whose projection has vanishing capacity. The condition $\lim_{z\to p} G(z, z_0)=0$ on the Green's function G of R and an accessible boundary point p guarantees that p is regular. *L. Sario* (Los Angeles, Calif.).

Kuramochi, Zenjiro. Dirichlet problem on Riemann surfaces. V. On covering surfaces. Proc. Japan Acad. **31**, 20–24 (1955).

A further discussion of types of covering surfaces introduced by the author in the third paper of this series.

L. Sario (Los Angeles, Calif.).

Kuramochi, Zenjiro. Harmonic measures and capacity of sets of the ideal boundary. I. Proc. Japan Acad. **30**, 951–956 (1954).

The author considers subsets of the ideal boundary of an open Riemann surface. He defines harmonic measures and capacities of such subsets, and discusses their properties and connections with Green's functions. *L. Sario.*

Kuramochi, Zenjiro. Harmonic measures and capacity of sets of the ideal boundary. II. Proc. Japan Acad. **31**, 25–30 (1955).

The paper deals with the behaviour of the Green's function in the neighborhood of the ideal boundary of an open Riemann surface R. A typical result is as follows. Suppose R is hyperbolic, and denote by $G(z, z_0)$ the Green's function on R, by $h(z)$ the conjugate of G. If R is suitably cut along level lines of h, then the function $w=f(z)=\exp(-G-ih)$ maps R onto the unit disk with a countable number of radial slits. The inverse function $z=f^{-1}(w)$ can be analytically continued along the radii $re^{i\theta}$ from $w=0$ to $|w|=1$, except for a set of values θ of angular measure zero.

L. Sario (Los Angeles, Calif.).

Bettermann, Rudolf. Riemannsche Gebiete. Schr. Math. Inst. Univ. Münster no. **7**, ii+69 pp. (1954).

In the first chapter the author considers a connected Hausdorff space $\mathfrak{B}$, which is a locally simple covering space over a Hausdorff space $\mathfrak{A}$, in which the defining neighbourhoods are connected. Boundary points of $\mathfrak{B}$ are defined by means of increasing sequences of subsets. The usual definitions of accessible and inaccessible boundary points, primitive boundary sets, and characteristic neighbourhoods of boundary points are adapted to this general case. A boundary point R of $\mathfrak{B}$ is called a branch point if the canonical mapping of $\mathfrak{B}$ into $\mathfrak{A}$ is several-valued in every neighbourhood of R. The maximum number of values of the canonical mapping is the same for all small neighbourhoods and is called the order of the branch point. In the second chapter the investigations are restricted to the special case $\mathfrak{A}=R_{2n}$. The author first considers branch points with a uniformization. By addition of a relatively open subset of the set of these branch points to $\mathfrak{B}$ the author obtains a domain which is called a complex manifold. A more general type of Riemannian domain is obtained by inclusion of nontranscendental branch points, i.e. branch points of finite order and with a neighbourhood in which the set of all boundary points of $\mathfrak{B}$ forms an analytic surface. The general Riemannian domain contains a $(2n-2)$-dimensional manifold consisting of non-transcendental branch points, and this manifold contains a $(2n-4)$-dimensional manifold outside which every branch point has a uniformization. A single-valued function on $\mathfrak{B}$ is called regular if it can be represented as root of a pseudo-polynomial in a neighbourhood of every point. It is proved that a function is regular on a Riemannian domain if it is continuous in the domain and regular except in the branch points, or if it is regular in the greatest analytic submanifold of the domain. In the last chapter the author gives an abstract definition of Riemannian domains and extends his principal results to these domains.

H. Tornehave (Copenhagen).

Koch, Karl. Die analytische Projektion. Schr. Math. Inst. Univ. Münster no. **6**, ii+79 pp. (1953).

Let G be a domain over the space of n complex variables, and f be a single-valued, non-constant, regular-analytic function on G; f is thus a mapping of G into the complex plane. An analytic projection P associated to the function f, in the abstract sense, is a mapping of G onto a Riemann surface $P(G)/f$ over the complex plane, such that the composite of the analytic projection P and the natural projection from the Riemann surface $P(G)/f$ into the complex plane coincides with the mapping f.

The author gives the following concrete construction of an analytic projection associated to any such domain G and function f. The point sets in which the function f takes a given value decompose into disjoint segments of analytic subvarieties of G of topological dimension $2n-2$. All points on the same segment of such a subvariety are equivalent; and a criterion is given for the equivalence of points on different segments. A point of $P(G)/f$ is an equivalence class of points of G. The major portion of the paper is devoted to proving that $P(G)/f$ may be given the structure of a Rie-

mann surface over the complex plane. The author shows that this particular analytic projection is associated to the function f in a natural manner, in the sense that whenever g is another such function on the same domain G, and f and g are functionally dependent (i.e., the matrix $(\partial f/\partial z_\nu, \partial g/\partial z_\nu)$, $\nu=1, \cdots, n$, is of rank less than two at each point), then the Riemann surfaces $P(G)/f$ and $P(G)/g$ are analytically homeomorphic. These two properties depend essentially upon the choice of the equivalence relation given. *R. C. Gunning* (Chicago, Ill.).

Hedtfeld, Karlheinz. Starre einfach zusammenhängende Holomorphiegebiete. Schr. Math. Inst. Univ. Münster no. 8, i+72 pp. (1954).

A domain G in the space of n complex variables is a star domain if the only analytic homeomorphism of G onto itself is the identity map. In the present paper the author constructs several classes of star domains of holomorphy, in the space of $n\geqq 2$ complex variables, which are of the topological type of a $2n$-cell (simply-connected in the author's terminology); there are of course none such in one complex variable.

There are two principal tools utilized in the construction. First is a theorem of W. Rothstein [Dissertation, Münster, 1935] concerning the restrictions which are imposed upon the possible analytic homeomorphisms between two domains when the domains are bounded in part by segments of analytic hypersurfaces. Second is the theory of analytic projection developed by K. Koch in the paper reviewed above. An example of the type of domain constructed is furnished by the analytic polyhedron G: $|z_1|<1, \cdots, |z_n|<1$, $|a_1z_1+\cdots+a_nz_n+c|<1$. Utilizing Rothstein's theorem, the author shows that an analytic automorphism must induce analytic homeomorphisms between the analytic projections of G by the functions $z_1, \cdots, z_n, a_1z_1+\cdots+a_nz_n+c$, in some order; by imposing suitable restrictions upon the coefficients, all automorphisms but the identity map will be excluded. The author constructs by this method infinitely many (indeed, an infinity of the order of the continuum) star polyhedrons which are not analytically equivalent to one another, but are all of the topological type of a $2n$-cell. The limitations of this method are also discussed. *R. C. Gunning* (Chicago, Ill.).

Spampinato, Nicolò. Sulle singolarità degli zeri di una funzione supercomplessa. Rend. Accad. Sci. Fis. Mat. Napoli (4) 20 (1953), 316–323 (1954).

The author studies the zeros of polynomials $F(\mu)$ of the hypercomplex variable $\mu=xu+yv$ where x and y are ordinary complex variables, u is the modulus, and $v^2=0$. Since $F(\mu)$ has the form $f(x)u+[f'(x)y+g(x)]v$, where $f(x)$ and $g(x)$ are ordinary complex polynomials, the zeros of F can be expressed in terms of zeros of f and g. For instance, if $g(x)$ vanishes identically then $F(\mu)$ has a zero if x is a double zero of $f(x)$ and y is arbitrary. The case where $g(x)$ has the form $(hx+k)f'(x)$, h and k constant, is given special consideration. *P. W. Ketchum* (Urbana, Ill.).

Theory of Series

Mikolás, Miklós. On a class of infinite products whose value can be expressed in closed form. Acta Sci. Math. Szeged 16, 58–62 (1955).

The author continues his previous work [Acta Sci. Math. Szeged 12, Pars A, 68–72 (1950); MR 12, 163] on the determination of a wide class of infinite products, including the Wallis product

$$\prod_{m=1}^{\infty}\frac{(2m)^2}{(2m-1)(2m+1)}=\frac{\pi}{2},$$

whose values can be expressed in finite form by means of the elementary functions. He now considers the expression

$$(*)\qquad \prod_{n=0}^{\infty}\frac{(A+nD+\frac{1}{2}(\nu-1)d)}{(A+nD)\cdots(A+nD+(\nu-1)d)}$$

for fixed nonvanishing complex numbers d, A, D, and for a fixed positive integer ν. It is shown that (*) converges if and only if none of the numbers $(A+\mu d)/D$ ($\mu=0, 1, \cdots, \nu-1$ and $\mu=\frac{1}{2}(\nu-1)$) is zero or a negative integer. If the product converges, it has the value

$$\Gamma\left(\frac{A+(\nu-1)d}{2D}\right)^{-\nu}\prod_{n=0}^{\nu-1}\Gamma\left(\frac{A+nd}{D}\right).$$

In some cases it is possible to reduce the value to an expression not involving the gamma function. Thus for $A=1$, $d=D=2$, and $\nu=2$ we obtain the Wallis product; and for $A=d=D=1$ and $\nu=2$ we have

$$\prod_{m=1}^{\infty}\frac{(m+\frac{1}{2})^2}{m(m+1)}=\frac{4}{\pi}.$$

E. F. Beckenbach (Los Angeles, Calif.).

Briggs, W. E., Chowla, S., Kempner, A. J., and Mientka, W. E. On some infinite series. Scripta Math. 21, 28–30 (1955).

The following two formulas are proved:

$$2\zeta(3)=\sum_1^\infty\frac{1}{n^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right),$$

$$\zeta(3)=\sum_1^\infty\frac{1}{n}\left\{\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\cdots\right\}.$$

[These formulas, the second in a slightly different form, had been stated as problems by M. S. Klamkin, Amer. Math. Monthly 59, 471 (1952).] *L. Carlitz.*

Berger, E. R. Bernoullische Zahlen, Potenzsummen und Stirlingsche Reihe. Z. Angew. Math. Mech. 35, 70–71 (1955).

Let $S_k(n)$ be defined by the relation $S_k(n+1)-S_k(n)=n^k$, $S_k(0)=0$. Almost as a direct consequence of the mean-value theorem the symbolic relation

$$S_k(n)=[(n+B)^{k+1}-B_{k+1}]/(k+1)$$

is obtained, in which the exponents of B are to be thought of as subscripts. The B_k are the Bernoulli numbers. A parallel derivation of Stirling's formula is given. *N. D. Kazarinoff* (Lafayette, Ind.).

Aljančić, S. Ein Verfahren zur Erzeugung von asymptotischen Entwicklungen. Bull. Soc. Math. Phys. Macédoine 5 (1954), 22–29 (1955). (Serbo-Croatian. German summary)

A function $c_\nu(x)$ is said to have an asymptotic expansion in the scale $\{q_\mu(x)\}$ if $c_\nu(x)\sim\sum_0^\infty \gamma_\mu(\nu)/q_\mu(x)$, $x\rightarrow\infty$, where $q_\mu(x)>0$, $q_\mu(x)/q_{\mu+1}(x)\rightarrow 0$, $x\rightarrow\infty$, $\mu=0, 1, \cdots$. Suppose $F(\alpha, x)=\sum_0^\infty c_\nu(x)\Phi_\nu(\alpha)$, $a\leqq\alpha\leqq b$ and $x\geqq 0$, where the c_ν have asymptotic expansions in the scale $\{q_\mu\}$. When does $F(\alpha, x)$ have an asymptotic expansion which may be found

by substituting the expansion for $c_\nu(x)$ in the series for $F(\alpha, x)$ and then interchanging the order of summation? Karamata [Acad. Serbe Sci. Publ. Inst. Math. 4, 69–88 (1952); MR 14, 372] has shown this to be true when $\Phi_\nu(\alpha)=e^{\nu\alpha i}$, while the author has done so when $\Phi_\nu(\alpha)=P_\nu(\cos\alpha)$ [ibid. 6, 115–124 (1954); MR 16, 691]. Here he gives the theorem: Let the sequence $\{a_\nu\}$ be completely monotone and let the elements of the sequence $\{b_\nu\}$ have asymptotic expansions in the scale $\{q_\mu\}$ with coefficients $P_\mu(\nu)$ where $P_\mu(\nu)$ is a polynomial in ν of degree at most s. Then for integral $k\geqq s$, if the kth difference of $b_\nu(x)$ is positive and decreasing as $x\to\infty$, the function $G(\alpha, x)=\lim_{r\to 1}\sum_0^\infty a_\nu b_\nu P_\nu^{(\lambda)}(\cos\alpha)r^\nu$ has an asymptotic expansion in the scale $\{q_\mu(x)\}$.
N. D. Kazarinoff (Lafayette, Ind.).

Fourier Series and Generalizations, Integral Transforms

Izumi, Shin-ichi, and Matsuyama, Noboru. Some trigonometrical series. I. J. Math., Tokyo 1, 110–116 (1953).

Let $-1<\alpha\leqq 1$. If

$$\sum_{j=1}^{n} ja_j=o(n^\alpha) \quad\text{and}\quad \sum_{j=n}^{\infty}|a_j-a_{j+1}|=o(n^{-(\alpha+1)/2}),$$

then the series $\sum a_j\sin jx$ converges uniformly in $0\leqq x\leqq\pi$. If $\sum a_j$ converges with either

$$\sum_{j=1}^{n} j\left(\sum_{k=j}a_k\right)=o(n^\alpha) \quad\text{and}\quad \sum_{j=n}^{\infty}|a_j-a_{j+1}|=o(n^{-(\alpha+1)/2})$$

or with

$$\sum_{j=1}^{n}\sum_{k=j}^{\infty}a_k=o(n^\alpha) \quad\text{and}\quad \sum_{j=n}^{\infty}|a_j-a_{j+1}|=o(n^{-(\alpha+1)/2}),$$

then the series $\sum a_j\cos jx$ converges uniformly in $0\leqq x\leqq\pi$. Applications are made to R_1 summability and to the integrability of the sum function of $\sum a_j\sin jx$. *P. Civin.*

Matsuyama, Noboru. Some trigonometrical series. II. J. Math., Tokyo 1, 117–127 (1953).

Let $f(x)\sim\sum_{n=1}^\infty a_n\cos nx$, and let $f_r(x)$ be the fractional integral of $f(x)$, $0<r\leqq 1$. A necessary and sufficient condition for the convergence of $\sum_{n=1}^\infty a_n/n$ is that $\int_{\to 0}^\pi f_r(t)t^{-r}dt$ exist in the Cauchy sense. If $g(x)\sim\sum_{n=1}^\infty a_n\sin nx$, $g(0+)=0$, and $\int_{\to 0}^\pi g(t)t^{-1}dt=0$, then a necessary and sufficient condition for the convergence of $\sum_{n=1}^\infty(\sum_{j=1}^n a_j)/n$ is the existence in the Cauchy sense of $\int_{\to 0}^\pi t^{-1}dt\int_t^\pi g(u)\cot\frac{1}{2}u\,du$.
P. Civin (Eugene, Ore.).

Izumi, Shin-ichi. Some trigonometrical series. III. J. Math., Tokyo 1, 128–136 (1953).

The author gives some conditions under which the convergence of $\int_{\to 0}^\pi f(x)dx$ implies the convergence of $\sum a_n/n$ or $\sum s_n/n$, where $f(x)$ is a sine or cosine series, a_n are its Fourier coefficients, and $s_n=a_1+\cdots+a_n$. He has proved converse theorems in a paper which has already been reviewed [Tôhoku Math. J. (2) 6, 73–77 (1954); MR 16, 240]. He also gives theorems connecting fractional integrals with sums $\sum a_n n^{-\alpha}$, $\sum s_n n^{-\alpha}$. [For older results of this kind cf. Bosanquet and Offord, Compositio Math. 1, 180–187 (1934).] *R. P. Boas, Jr.* (Evanston, Ill.).

Izumi, Shin-ichi. Some trigonometrical series. IV. Tôhoku Math. J. (2) 5, 18–21 (1953).

Examples are given of cosine series $\sum a_n\cos nx$ with $na_n\to 0$ which (1) converge at $x=0$, but not in a neighborhood of $x=0$, and (2) which converge for all x, with sum discontinuous at $x=0$. *P. Civin* (Eugene, Ore.).

Izumi, Shin-ichi. Some trigonometrical series. V. Tôhoku Math. J. (2) 5, 29–33 (1953).

Let $f(x)\in L[0, 1]$ and $f(x+1)=f(x)$. Let

$$F_n(x)=n^{-1}\sum_{j=1}^{n}f(x+jn^{-1})$$

be the Riemann sum of $f(x)$. Say $f(x)=f(-x)$ has Fourier cosine coefficients $n^{-1/2}(\log n)^{-\alpha}$. Then as $n\to\infty$, $F_n(x)$ diverges almost everywhere if $\alpha\leqq\frac{1}{2}$ and converges almost everywhere if $\alpha>\frac{1}{2}$. A similar result is obtained for a class of cosine series containing the above example. *P. Civin.*

Satô, Masako. Uniform convergence of Fourier series. III. Proc. Japan Acad. 30, 809–813 (1954).

[For parts I and II see same Proc. 30, 528–531, 698–701 (1954); MR 16, 692, 919.] Generalizations of the familiar Dini-Lipschitz test for uniform convergence. For instance, if $f(x+h)-f(x)=o(\log|h|^{-\alpha})$, $0<\alpha<1$, and if the Fourier coefficients are $O(n^{-1}\exp(\log n)^\alpha)$, then $s_n(x_n)\to f(x)$ uniformly for all $x_n\to x$, where s_n is the nth partial sum of the Fourier series (uniform convergence at x). [Cf. S. Izumi and G. Sunouchi, Tôhoku Math. J. (2) 3, 298–305 (1951); MR 13, 838.] *W. W. Rogosinski.*

Žak, I. E. On a theorem of Lévy on absolute convergence of Fourier series. Uspehi Mat. Nauk (N.S.) 10, no. 1(63), 107–112 (1955). (Russian)

A theorem of Wiener and P. Lévy [see, e.g., A. Zygmund, Trigonometrical series, Warszawa-Lwów, 1935, Theorem 6.51] is stated and proved for functions of 2 variables. However, all theorems of this kind are special cases of a theorem of I. M. Gel'fand [Mat. Sb. N.S. 9(51), 3–24 (1941), Satz 20; MR 3, 51]. The general form of Wiener's theorem is the following. Let G be a compact Abelian group, with character group G^*. Let f be a complex function on G of the form

$$(1)\qquad f(x)=\sum_{\chi\in G^*}\alpha_\chi\chi(x),\quad \sum_{\chi\in G^*}|\alpha_\chi|<\infty.$$

Let Φ be a function analytic on the set $f(G)$. Then $g(x)=\Phi(f(x))$ also has the form (1). This assertion follows at once from the theorem of Gel'fand cited above. For, let $\mathfrak{A}$ be the algebra of all complex functions on G of the form (1). This algebra is isomorphic to the $\mathfrak{L}_1$-algebra of G^* and its space of maximal ideals is accordingly G itself, with the original topology of G. The generalized Wiener theorem now follows upon applying Gel'fand's theorem to this algebra $\mathfrak{A}$. The classical case is that in which G is the circle group, and the case treated by Žak is that in which G is the 2-dimensional torus group. *E. Hewitt* (Princeton, N. J.).

Zeller, Karl. Über Konvergenzmengen von Fourierreihen. Arch. Math. 6, 335–340 (1955).

Let E be a set of type F_σ on the half-open interval $I=[0, 2\pi)$. Then there exists a 2π-periodic function f in L_1 whose Fourier series converges at each point of E and diverges unboundedly at each point of $I-E$. The author proves this result with the aid of Kolmogoroff's example of a function whose Fourier series diverges everywhere [C. R. Acad. Sci. Paris 183, 1327–1328 (1926)]. *G. Piranian.*

Heywood, P. On the integrability of functions defined by trigonometric series. II. Quart. J. Math., Oxford Ser. (2) 6, 77–79 (1955).

[For part I see same J. (2) 5, 71–76 (1954); MR 16, 30.] The author proves that if λ_n is ultimately positive, if $\frac{1}{2}\lambda_0+\sum_1^\infty \lambda_n=0$, and if $f(x)=\frac{1}{2}\lambda_0+\sum_1^\infty \lambda_n \cos nx$, then $|f(x)/x|$ is integrable if and only if $\sum \lambda_n \log n$ converges. [This seems to be equivalent, by partial summation, to the theorem on p. 112 of Zygmund, Trigonometrical series, Warsaw-Lwów, 1935.] *R. P. Boas, Jr.*

Izumi, Shin-ichi, and Sato, Masako. Integrability of trigonometrical series. I. Tôhoku Math. J. (2) 6, 258–263 (1954).

Let $f(x)\sim\sum a_n \cos nx$, $0<\alpha<1$. The authors show that if $x^{\alpha-1}f(x)$ is absolutely integrable, $\sum a_n/n^\alpha$ converges, while if $\sum a_n/n^\alpha$ converges absolutely, $\int_{+0} f(t)t^{\alpha-1}dt$ exists. They also give two more complicated theorems in which the absolute convergence or absolute integrability in the hypothesis is weakened. *R. P. Boas, Jr.* (Evanston, Ill.).

Aljančić, S., Bojanić, R., et Tomić, M. Deux théorèmes relatifs au comportement asymptotique des séries trigonométriques. Srpska Akad. Nauka. Zb. Rad. 43. Mat. Inst. 4, 15–26 (1955). (Serbo-Croatian. French summary)

A positive function $L(t)$ is called slowly increasing if $L(\lambda t)/L(t)\to1$ as $t\to\infty$, for each λ. The authors show that (i) if $L(t)$ is the product of two monotonic slowly increasing functions, then for $0<\alpha<2$, $x\to0+$,

$$\sum_{n=1}^{\infty} L(n)n^{-\alpha} \sin nx\sim\frac{\pi}{2\Gamma(\alpha) \sin \alpha\pi/2}x^{\alpha-1}L(x^{-1});$$

(2) if $L(t)$ is slowly increasing, convex, and tends to 0, then

$$\sum L(n) \sin nx\sim x^{-1}L(x^{-1}).$$

These generalize known results in which, respectively, (1) $L(n)\sim1$ and (2) $L(n)$ is monotonic [see Heywood, J. London Math. Soc. 29, 373–378 (1954); MR 15, 952; and Zygmund, Trigonometrical series, Warsaw-Lwów, 1935, p. 114]. *R. P. Boas, Jr.* (Evanston, Ill.).

Helson, Henry. On a theorem of F. and M. Riesz. Colloq. Math. 3, 113–117 (1955).

A new and much simpler proof of the familiar theorem by F. and M. Riesz [see A. Zygmund, Trigonometrical series, Warszawa-Lwów, 1935, p. 158]. If $\mu(x)$ is of bounded variation in $\langle 0, 2\pi\rangle$ and $\int_0^{2\pi} e^{-inx}d\mu(x)=0$ for $n<0$, then $\mu(x)$ is absolutely continuous. The proof is based on the following lemma: Let $\mu(x)$ be 'singular' and $\nu(x)=\int_0^x|d\mu(t)|$. Then there exists a function $\phi(z)$, regular and satisfying $0<|\phi(z)|<1$ in $|z|<1$, such that $\lim_{r\to1}\phi(re^{ix})=0$ almost everywhere with respect to ν; i.e., the exceptional set carries all the mass of the measure ν induced by $\nu(x)$.
W. W. Rogosinski (Newcastle-upon-Tyne).

Lorch, Lee. The limit of a certain integral containing a parameter. Amer. Math. Monthly 62, 433–434 (1955).

The author proves that

$$\frac{1}{n}\int_0^{\pi/2}\left|\frac{\sin (2n+1)\theta}{2 \sin \theta} \cot \theta-\frac{(2n+1) \cos (2n+1)\theta}{2 \sin \theta}\right| \sin 2\theta d\theta$$
$$=\frac{4}{\pi}+O\left(\frac{\log n}{n}\right).$$

This is estimated by integrating the absolute value of each integrand separately. *S. Izumi* (Tokyo).

Berkovitz, Leonard D., and Gosselin, Richard P. Restricted summation and localization of double trigonometric series. Duke Math. J. 22, 243–251 (1955).

The object of this paper is to present a theory of localization for double trigonometric series involving the usual type of neighbourhood and using double index summation methods, i.e. restricted Riesz summation with kernel $(1-t^2)^\lambda$. This theory has the advantage that it can be easily extended to the case of higher dimension [cf. V. L. Shapiro, Amer. J. Math. 75, 347–357 (1953); MR 14, 974].

Let $\sum a_{\mu\nu}$ be a double series and let

$$\sigma_{R,T}^{(\alpha,\beta)}=\sum_{|\mu|\leqq R,\,|\nu|\leqq T} a_{\mu\nu}(1-\mu^2/R^2)^\alpha(1-\nu^2/T^2)^\beta,$$

where $\alpha>0$ and $\beta>0$. If $\sigma_{R,T}^{(\alpha,\beta)}\to l$ as R, $T\to\infty$, then it is said that the series $\sum a_{\mu\nu}$ is restrictedly (Riesz) summable (R, α, β) to l. The formal multiplication theorem reads as follows: Let $T=\sum a_{mn}e^{i(mx+ny)}$ be a trigonometric series such that

(i) $a_{mn}=o((|m|+1)^\gamma(|n|+1)^\delta)$ $\quad(\gamma>-1, \delta>-1, \gamma+\delta\geqq-1)$,

(ii) $a_{mn}=O((|m|+1)^{-3\beta'-3}(|n|+1)^{-3\beta'-3})$ $\quad(\beta=\gamma+\delta+1)$,

where β' is the smallest integer greater than or equal to β, and let $\lambda(x, y)\sim\sum\alpha_{mn}e^{i(mx+ny)}$, which together with all of its continuous derivatives up to and including those of order $3\beta'$ is zero for (x, y) belonging to a set E in $(0, 2\pi)\times(0, 2\pi)$; then the formal product series

$$\sum A_{mn}e^{i(mx+ny)},\quad A_{mn}=\sum_{p,q=-\infty}^{\infty} a_{p,q}\alpha_{m-p,n-q}$$

is restrictedly equisummable (R, β, β) with the series

$$\lambda(x, y)\sum a_{mn}e^{i(mx+ny)}$$

uniformly for (x, y) in E.

By this theorem, the authors prove the following localization theorem containing the Riemann formula for double trigonometric series: Let $\lambda(x, y)=\varphi(x)\psi(y)$, where $\varphi(x)$ and $\psi(y)$ are periodic functions with period 2π such that φ and ψ are 0 outside the closed intervals X and Y, respectively, and are 1 inside the intervals X' and Y', strictly contained in X and Y, respectively, and that φ and ψ have continuous derivatives of order $K+3\beta'+3$ and $L+3\beta'+3$, respectively. If $F(x, y)$ is a continuous function gotten by the K and L time integration of T, with respect to x and y, then

$$\sum_{m,n=-M,-N}^{M,N} a_{mn}e^{i(mx+ny)}-(-1)^{K+L}\pi^{-2}\int_0^{2\pi}\int_0^{2\pi} F(u, v)$$
$$\times\lambda(u, v)D_M^{(K)}(x-u)D_N^{(L)}(y-v)dudv$$

is restrictedly summable (R, β, β) to zero, uniformly in the rectangle $X'\times Y'$, where $D_M(x)$ is the Mth Dirichlet kernel and $D_M^{(K)}(x)$ is the Kth derivative of $D_M(x)$. *S. Izumi.*

Erdös, P., and Gál, I. S. On the law of the iterated logarithm. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. 58 = Indag. Math. 17, 65–76, 77–84 (1955).

Le but de cette Note est de démontrer le résultat suivant: Soit $n_1, n_2, \cdots, n_k, \cdots$ une suite croissante de nombres positifs satisfaisant à la condition $n_{k+1}/n_k\geqq q>1$

($k=1, 2, \cdots$). Dans ces conditions

$$\limsup \frac{|\sum_{k=1}^{N} \exp 2\pi i n_k x|}{(N \log\log N)^{1/2}} = 1.$$

L'inégalité $\limsup \leq 1$ avait été démontrée par Salem et Zygmund [Bull. Sci. Math. (2) **74**, 209–224 (1950); MR **12**, 605] pour le cas des n_k entiers, dans les conditions plus générales où chaque exponentielle est multipliée par un coefficient a_k satisfaisant aux conditions de Kolmogoroff:

$$A_N = \sum_1^N a_k^2 \to \infty, \quad a_n = o\left(\frac{A_N}{\log\log A_N}\right)^{1/2},$$

l'inégalité s'écrivant dans ce cas:

$$\limsup \frac{|\sum_1^N a_k \exp 2\pi i n_k x|}{(A_N \log\log A_N)^{1/2}} \leq 1.$$

Les auteurs de la présente Note affirment, sans donner de démonstration, pouvoir étendre leur théorème au cas des coefficients quelconques, qui est plus compliqué (c'est à dire transformer l'inégalité précédente en égalité: la premier formule de la p. 67 de leur note est sans doute le résultat d'une erreur de plume). *R. Salem* (Paris).

Calderón, A. P., and Zygmund, A. Singular integrals and periodic functions. Studia Math. **14** (1954), 249–271 (1955).

The general theory of the operators

$$(1)\quad F^*(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(y)K(x-y)dy; \quad K(t) = \frac{1}{t}, \; -\infty < x < \infty,$$

$$(2)\quad f^*(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)K^*(x-y)dy; \quad K(t) = \tfrac{1}{2}\cot\frac{t}{2}, \quad -\pi \leq x \leq \pi,$$

where the integrals are taken in the principal-value sense, was evolved by M. Riesz, after the case $F \in L_2(-\infty, \infty)$, $f \in L_2(-\pi, \pi)$ had been dealt with by Hardy and other writers. His work was continued by other mathematicians, e.g. by Privaloff and Zygmund; and recently also by Walsh and Elliott. The operator (2) relates to periodic functions and plays a considerable part in the theory of Fourier series. Again in his first proof of the properties of (1), Riesz had deduced them from those of (2).

In a former paper [Acta Math. **88**, 85–139 (1952); MR **14**, 637] the authors had extended (1) to the case when the kernel K is a rather general function of k variables in the Euclidean space E^k, and had applied results to potential theory. In the present paper they discuss the extension of (2). But, differently from Riesz, they deduce the case (2) from (1): they introduce the periodic kernel

$$K^*(x) = K(x) + \sum_{\nu=1}^{\infty} \{K(x-x_\nu) - K(-x_\nu)\} \quad (x \in E^k),$$

where $\{x_\nu\}$ is the sequence of all lattice points generated by k independent vectors in E^k (a special case for $k=2$ is $K^*(x) = \wp(x)$), and replace the range E^k of integration which occurs in the extension of (1) by the "fundamental cube" $R \subset E^k$: $|\xi_j| \leq \frac{1}{2}$, $j=1, 2, \cdots, k$. In this way they deduce seven basic Theorems on $f^*(x)$ ($x \in R$) in §§2 and 3. In §§4 and 5 they deal with the series that is conjugate, with respect to the kernel K^*, to the Fourier series of a periodic function $f(x)$ ($x \in R$) and prove the following theorem. Theorem 8. If $f^*(x)$ is integrable (in particular if $f^* \in L^p(R)$, $p>1$), then the Fourier series of $f^*(x)$ has coefficients $c_m^* = c_m \gamma_m$, where the c_m are the Fourier coefficients of $f(x)$ and $\gamma_m = \int_R K^*(y) \exp\{-2\pi i(m_1 y_1 + \cdots + m_k y_k)\}dy$. Here m is the vector $m_1, \cdots, m_k$ in E^k, and every m_j may be any integer.

In §6 results are applied to a special kernel. In §7 the Lipschitz condition $|f(x+h) - f(x)| < C|h|^\alpha$ ($0 < \alpha \leq 1$) is treated: the classical result due to Privaloff is extended to the k-dimensional case; so is the result for the crucial case $\alpha = 1$; it had been obtained by Zygmund for $k=1$ by means of the notion of a smooth function previously, as the Privaloff theory fails here. In §8 a result is given on discrete analogs of the Hilbert transform, i.e. on extensions of some classical results on the sequence $\tilde{x}_m = \sum_{n=-\infty}^{\infty} x_n(m-n)^{-1}$ ($m=0, \pm 1, \pm 2, \cdots; n \neq m$). In §9 a theorem is proved on the Fourier series of a bounded function, and in §10 remarks are made concerning smooth functions. *H. Kober.*

Ehrenpreis, L., and Mautner, F. I. Some properties of the Fourier transform on semi-simple Lie groups. I. Ann. of Math. (2) **61**, 406–439 (1955).

Soit G le groupe des transformations conformes du disque unité (i.e. le groupe quotient de SL(2, R) par son centre). Soit K le sous-groupe des rotations autour de 0. Une fonction sphérique sur G est une fonction constante sur les doubles classes KgK. Soit H l'espace hilbertien des fonctions $a(z) = a(\theta)$ ($z = e^{2\pi i\theta}$) sur le cercle unité, avec

$$\|a\|^2 = \int_0^1 |a(\theta)|^2 d\theta < +\infty.$$

Pour tout nombre complexe s, on pose

$$[U(g, s)a](z) = |dgz/dz|^s a(gz).$$

Alors $g \to U(g, s)$ est une représentation continue de G dans H, unitaire si et seulement si $\Re s = \frac{1}{2}$. Si f est une fonction sur G, $\int f(g)U(g, s)dg = \mathfrak{F}(s)$ est une "transformée de Fourier" opératorielle de f. Par rapport à la base orthonormale des $e^{2\pi i n\theta}$, les coefficients matriciels de $\mathfrak{F}(s)$ sont, pour f sphérique, tous nuls sauf

$$F(s) = \int f(g)(U(g, s)1, 1)dg = \int f(g)\varphi(g, s)dg,$$

où

$$\varphi(g, s) = \int_0^1 |dg\theta/d\theta|^s d\theta.$$

Dans cet article, consacré principalement aux fonctions sphériques, $f(g) \to F(s)$ est donc considéré comme la transformation de Fourier (scalaire). Si $f(g)$ appartient à l'ensemble A_1 des fonctions intégrables sphériques, $F(s)$ est analytique pour $0 \leq \Re s < 1$, continue pour $0 \leq \Re s \leq 1$, vérifie $F(s) = F(1-s)$, et $\lim_{t\to\pm\infty} F(\sigma + it) = 0$ uniformément en σ pour $0 \leq \sigma \leq 1$. Par analogie avec l'espace $\mathfrak{L}$ de L. Schwartz, on désigne par S l'espace vectoriel topologique des fonctions sphériques indéfiniment différentiables qui sont à décroissance rapide ainsi que certains de leurs dérivées (construites à l'aide du laplacien Δ sur G). Les auteurs caractérisent l'espace vectoriel topologique $\hat{S}$ des transformées de Fourier des fonctions de S par des conditions de décroissance sur les dérivées de $F(s)$; et $f(g) \to F(s)$ est un isomorphisme de S sur $\hat{S}$. Ainsi, le "théorème de Wiener" est faux si on se limite par exemple à envisager la transformée de Fourier $F(s)$ sur la droite $\Re s = \frac{1}{2}$, i.e. si on exclut celles des représentations de G envisagées qui ne sont pas unitaires. Mais soit $f \in A_1$; l'idéal fermé engendré par f est A_1 si par exemple $F(s)$ est analytique et non nulle dans une bande $-\delta < \Re s < 1 + \delta$

($\delta>0$) et vérifie certaines conditions quand $s\to\pm i\infty$. Soit $F(s)$ une fonction de variable complex; pour qu'elle soit transformée de Fourier d'une $f(g)$ indéfiniment différentiable à support compact, il faut et il suffit que $F\in S$ (dans la bande $0\leqq\Re s\leqq 1$) et que $F(s)$ soit entière de type exponentiel ("théorème de Paley-Wiener"). Enfin, les auteurs caractérisent les transformées de Fourier des "distributions à support compact", définies en considérant le dual de l'espace des fonctions sphériques indéfiniment différentiables, topologisé par la convergence compacte des $\Delta^m f$. Le résultat est encore analogue au résultat abélien. Les outils principaux sont: 1) certains résultats de Bargmann [Ann. of Math. (2) **48**, 568–640 (1947); MR **9**, 133]; 2) la formule de Plancherel pour SL(2, R) due à Harish-Chandra [Proc. Nat. Acad. Sci. U. S. A. **38**, 337–342 (1952); MR **13**, 820]; 3) des calculs explicites. Des extensions et des applications sont promises pour les parties II et III.

Quelques errata: p. 419, l. 1 du bas, remplacer 2 par 2ζ; p. 421, l. 6, remplacer $s(1-s)$ par s; l. 7 du bas, remplacer $-\pi t$ par $\pm\pi t$; p. 422, l. 3, multiplier par 1/2; p. 435, l. 14 du bas, remplacer 2 par $\sqrt{2}$. *J. Dixmier* (Paris).

Ilieff, Ljubomir. Trigonometrische Integrale, die ganze Funktionen mit nur reellen Nullstellen darstellen. Bŭlgar. Akad. Nauk. Izv. Mat. Inst. **1**, no. 2, 147–153 (1954). (Bulgarian. Russian and German summaries)

If $f_0(t)$ is nonnegative and $\int_0^a f_0(t)\cos tz\,dt$ has only real zeros; if $x(t)$ is real and even, $x(a)=0$, and (P) $x'(it)$ is a polynomial with only real zeros or an entire function which is a limit of such polynomials; if $t=t(x)$ is the inverse of $x(t)$, and $f_0(t)=\phi_0(x)$, $f_1(t)=\phi_1(x)=\int_0^x\phi_0(u)\,du$; then $\int_0^a f_1(t)\cos tz\,dt$ has only real zeros. It follows that, if $x(t)$ satisfies the conditions above and $x(0)>0$, then

$$\int_0^a x(t)^\lambda \cos tz\,dt$$

has only real zeros for $\lambda>-1$; and if $x(t)$ is real and nonnegative and satisfies (P), then $\int_0^a e^{-x(t)}\cos zt\,dt$ has only real zeros. *R. P. Boas, Jr.* (Evanston, Ill.).

Obrechkoff, N. Sur quelques représentations intégrales de fonctions en liaison des équations différentielles. Bŭlgar. Akad. Nauk. Izv. Mat. Inst. **1**, no. 2, 3–33 (1954). (Bulgarian. Russian and French summaries)

The author investigates the inversion and representation theory of the transform $f(x)=\int_0^\infty\Phi(xt)d\omega(t)$, when

$$\Phi(x)=\int_0^\infty\int_0^\infty \exp\{-t-u-x(tu)^{-1}\}t^{\alpha-1}u^{\beta-1}dudt.$$

R. P. Boas, Jr. (Evanston, Ill.).

Lakshmanarao, S. K. Gegenbauer transforms. Math. Student **22** (1954), 161–165 (1955).

The convolution property for the Gegenbauer integral transformation is derived here. This generalization from the convolution for the Legendre integral transformation was carried out independently by the author and by S. D. Conte [Quart. J. Math. Oxford Ser. (2) **6**, 48–52 (1955); MR **16**, 922]. *R. V. Churchill* (Ann Arbor, Mich.).

Brauer, George. A note on Euler transforms. Amer. Math. Monthly **62**, 432–433 (1955).

Let E_λ be the class of functions $f(x)$ of the form

$$f(x)=\int_0^\infty (x+t)^{-\lambda}d\phi(t),$$

where $\phi(t)$ is monotone nondecreasing ($0\leqq t<\infty$). It is shown that if $f\in E_\lambda$ ($\lambda>1$) and if $\phi(t)$ is absolutely continuous with a derivative which satisfies the conditions (i) $\phi'(t)$ is monotone nondecreasing ($0\leqq t<\infty$), (ii) $\phi(t)=O(t^{\lambda-1})$ as $t\to+\infty$, then $f\in E_{\lambda-1}$. *I. I. Hirschman, Jr.*

van Kampen, N. G. Completeness of stationary scattering states. I. Physica **21**, 127–136 (1955).

Let $\phi(r)\in L^2(0,\infty)$ and let $\eta(k)$ be a given function on $0<k<\infty$. A sufficient condition that for any ϕ there exist in $L^2(0,r)$ a function $C(k)$ such that

$$\phi(r)=\int_0^\infty C(k)\sin(kr+\eta(k))dk$$

is that $S(x)=\exp(2i\eta(x))$ have a holomorphic and bounded analytic continuation in the upper half complex z-plane. The author shows this condition is not necessary [thus disproving earlier statements by Heisenberg and Hu] and gives necessary and sufficient conditions. He observes however that the above condition is necessary if the causality condition is postulated. *N. Levinson.*

Burr, E. J. Sharpening of observational data in two dimensions. Austral. J. Phys. **8**, 30–53 (1955).

Let $f(x,y)$ denote a true statistical frequency distribution in two dimensions, or the true distribution of intensity, over the xy-plane, of an observed phenomenon such as light or gravity. Let $g(x,y)$ represent the observed value of that distribution. If $h(x-x_0,y-y_0)$ is the observed distribution at (x,y) when the true distribution consists of a unit point source at (x_0,y_0), then

$$(*)\qquad g(x,y)=\int_{-\infty}^\infty\int_{-\infty}^\infty h(x-x',y-y')f(x',y')dx'dy'.$$

The integral represents the convolution of the functions f and h. Several experimental techniques are cited in which the function h can be described. As an example, if a source of light is scanned by a photoelectric cell behind a circular aperture, $h(x,y)$ may be a known positive constant inside a fixed circle and zero outside that circle. The problem of sharpening observational data is the problem of solving the integral equation (*) for $f(x,y)$ when the functions g and h are given. Thus the purpose of this procedure is to eliminate those errors resulting from that observational technique which controls the kernel $h(x,y)$. It is pointed out that when the observed values $g(x,y)$ are uncertain then there is no unique solution for f until solutions are stabilized by excluding rapidly oscillating functions. Two-dimensional Fourier transforms of f, g and h are denoted by capital letters. From equation (*) it follows that $G(u,v)=H(u,v)F(u,v)$, under conditions that are specified in the paper. Thus $F(u,v)$, and hence $f(x,y)$, can be solved for, at least formally. When the functions depend only on the polar coordinate r, the double Fourier transforms reduce to simple Hankel transforms, and the integral equation corresponding to (*) is given. Appropriate tables of Fourier and Hankel (Fourier-Bessel) transforms are given. Inverse kernels, the inverse transforms of $[H(u,v)]^{-1}$, and their approximations, are discussed. The shortcomings of the methods of using Fourier transforms are explained. The techniques of solving the equation (*) for f by representing f and g by polynomials are reviewed at some length. Finally, the author presents a rough guide for selecting the best method for a given problem. *R. V. Churchill* (Ann Arbor, Mich.).

***Puig Adam, P. Les systèmes linéaires rétroactifs en chaine et les fractions continues.** Les machines à calculer et la pensée humaine, pp. 495–513; discussion, 514. Colloques internationaux du Centre National de la Recherche Scientifique, no. 37. Centre National de la Recherche Scientifique, Paris, 1953. 2000 francs.

A physical system is described by a transformation $y(t)=F[x(t)]$ between the input function of time $x(t)$ and the output $y(t)$. The transformation F is assumed to be linear, continuous, and invariant under translations of time. The system is defined as stable if F is a bounded transformation. If F is expressible as an integral transform, with kernel function $K(t-\tau)$ and integrated with respect to τ from 0 to t, then stability is equivalent to the absolute integrability of $K(\tau)$ on the interval 0 to ∞. A necessary condition for stability is that the Laplace transform of $K(t)$ be bounded in the right half-plane. For physical systems in cascade the Laplace transform of the overall K is a continued fraction in the transforms of the individual K's. The Nyquist diagram can be constructed from the continued fraction by a sequence of elementary inversions and translations. Particular cases of ladder networks are worked out and the limiting case of a transmission line is discussed. A numerical analysis of an empirical curve for the response of nerve is made and interpreted. *P. W. Ketchum* (Urbana, Ill.).

Polynomials, Polynomial Approximations

Mitrović, Dusan. Conditions graphiques pour que toutes les racines d'une équation algébrique soient à parties réelles négatives. C. R. Acad. Sci. Paris **240**, 1177–1179 (1955).

Consider a polynomial equation of the nth degree, $\sum a_iZ^i=0$, where the a's are real and $a_n=1$. If a_1 and a_0 are regarded as variables ξ, η, then the locus of points (ξ, η) which satisfy the equation for any pure imaginary Z is a curve in the (ξ, η)-plane, which can be plotted readily. The required conditions (for the roots to have negative real parts) are that as $t=Z/i$ increases from 0 to ∞ the corresponding point on the curve will alternately cross the lines $\eta=a_0$ and $\xi=a_1$, beginning with the former, for a total of $n-1$ intersections. *P. W. Ketchum* (Urbana, Ill.).

Teicher, Henry. Reducibility of positive type polynomials. Proc. Amer. Math. Soc. **6**, 195–201 (1955).

A polynomial is said to be of positive type if all its coefficients are non-negative. The set of all such polynomials is denoted by T_+. Evidently T_+ is closed under multiplication but not factorization. A polynomial is irreducible in T_+ either if it is irreducible over the real field or else if in any factorization at least one factor does not belong to T_+. It is of interest to have criteria for irreducibility in T_+. When the degree of the polynomial is 2 or 3 such a criterion can readily be given. But already in the case of a quartic the situation is difficult, and the exact solution depends on the location of a root of the polynomial. The first step towards a general solution is to examine the nature of those polynomials in T_+ which have a factor of the form $x^2-Ax+A^2\epsilon$ ($A>0$, $\epsilon>\frac{1}{4}$). This is done in detail and a number of general theorems are presented bearing on this question. The paper includes also several results dealing with special cases.

W. Ledermann (Manchester).

Tomić, Boško S. Sur une classe des polynômes et sur les intégrales s'y rattachant. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **9**, 229–243 (1954). (Serbo-Croatian summary)

Put

$$A_\nu^n(a)=\sum_{i=0}^{\nu}(-1)^i\binom{n+1}{i}(a+\nu-i)^n,$$

$$B_\nu^n(a)=\sum_{i=0}^{\nu}(-1)^i\binom{n}{i}(a+\nu-i)^n,$$

so that $(d/da)B_\nu^n(a)=nA_\nu^{n-1}(a)$. The polynomial $A_\nu^n(a)$ was introduced by Nielsen in his Traité élémentaire des nombres de Bernoulli [Gauthier-Villars, Paris, 1923] and has been discussed in a recent paper by the present author [same Glasnik **9**, 97–108 (1954); MR **16**, 585]. In the present paper the author continues his study of $A_\nu^n(a)$ and $B_\nu^n(a)$. It is shown for example that

$$e^{(1-x)az}(1-xe^{(1-x)z})^{-1}=\sum_{n,\nu=0}^{\infty}\frac{1}{n!}B_\nu^n(a)z^nx^\nu.$$

Many relations involving $A_\nu^n(a)$ and $B_\nu^n(a)$ are obtained. For example

$$A_\nu^n(a)=A^n_{n-\nu}(1-a),\quad B^n_{n-\nu-1}(a),\quad B_\nu^n(1-a)=n!,$$

$$\sum_{i=0}^{\nu}A_i^n(a)=B_\nu^n(a),\quad B_n^{2n}(0)=\tfrac{1}{2}(2n)!.$$

Of trigonometric integrals connected with $B_\nu^n(a)$ we cite the formula

$$\int_{-\infty}^{\infty}\frac{\sin(nt)\sin^n t}{t^{n+1}}\cos(2xt)dt=\frac{\pi}{n!}B_\nu^n(-x+n-\nu).$$

L. Carlitz (Durham, N. C.).

Good, I. J. A new finite series for Legendre polynomials. Proc. Cambridge Philos. Soc. **51**, 385–388 (1955).

The author gives, with an eye to numerical applications, several instances of the formula

$$\int_0^{2\pi}f(\theta)d\theta=2\pi t^{-1}\sum_{r=0}^{t-1}f(2\pi r/t),$$

$t\geqq m$, where $f(\theta)$ is a trigonometric polynomial of order not exceeding m. These include a lattice-point problem; a sum of squares of the terms of a binomial series; and (the series of the title) the formula

$$P_N(x)=t^{-1}\sum_{r=0}^{t-1}\{x+(x^2-1)^{\frac{1}{2}}\cos(2\pi r/t)\}^N,$$

where $t>N$. *R. P. Boas, Jr.* (Evanston, Ill.).

Obrechkoff, N. Sur le développement des fonctions analytiques suivant des polynomes orthogonaux. C. R. Acad. Bulgare Sci. **7** (1954), no. 2, 5–8 (1955). (Russian summary)

This is a preliminary report on the author's study of polynomials which are orthogonal on a contour C with a weight $F(z)$ that is regular on C but has singular points in the domain D bounded by C. Ordinary orthogonal polynomials on a real interval (a, b) with weight $d\psi(t)$ can be subsumed by taking $F(z)=\int_a^b (z-t)^{-1}d\psi(t)$. The author sketches his methods, which depend on an asymptotic formula for the polynomials. He gives, by way of illustration, some formulas for the case $F(z)=e^{1/z}$.

R. P. Boas, Jr. (Evanston, Ill.).

Berman, D. L. Approximation of continuous functions by S. N. Bernštein's interpolation polynomials. Dokl. Akad. Nauk SSSR (N.S.) **101**, 397–400 (1955). (Russian)

S. Bernstein [Collected works, vol. 2, Izdat. Akad. Nauk SSSR, Moscow, 1954; MR **16**, 433] proved that if the polynomial $A_n(f, x)=\sum_{j=1}^n B_j l_j(x)$ is obtained by means of Lagrange interpolation for the Chebyshev nodes $x_j=\cos\theta_j$, $\theta_j=(2j-1)\pi/2n$, $j=1, \cdots, n$, and with $B_j=f(x_j)$ for all j not divisible by $2p$, while

$$B_{2kp}=f(x_{2kp-2p+1})-f(x_{2kp-2p+2})+\cdots+f(x_{2kp-1}),$$

then $A_n(f, x)\to f(x)$ uniformly. The author continues his investigation [same Dokl. (N.S.) **85**, 461–464 (1952); MR **14**, 165] of the degree of approximation. His main result (somewhat simplified) is that

$$|A_n(f, x)-f(x)|\leqq C_1 p\omega(4p\pi n^{-1})+C_2 pn^{-1}\int_{1/n}^{1/p} x^{-2}\omega(x)dx,$$

where $\omega(h)$ is the modulus of continuity of $f(x)$.

G. G. Lorentz (Detroit, Mich.).

Freud, G. Über orthogonale Polynome. Acad. Math. Acad. Sci. Hungar. **5**, 291–298 (1954). (Russian summary)

The $p_n(x)$ are the orthonormal polynomials with respect to a non-negative weight function $w(x)$ in $\langle -1, 1\rangle$. In an earlier paper [same Acta **3**, 83–88 (1952); MR **14**, 467] the author proved that the corresponding Fourier series of an $f\in L^2$ (with respect to w) is strongly $(C, 1)$-summable at almost all points x where (*) $\sum_0^{n-1} p_k^2(x)=O(n)$. Earlier results point to the (improved) conjecture that (*) holds for almost all x where $w(x)>0$. In this direction it is now proved: I. Let $W(\vartheta)=w(\cos\vartheta)\sin\vartheta$ in $\langle 0, \pi\rangle$, and 0 otherwise. If, for small h,

$$\int_0^\pi \frac{|W(\vartheta+h)-W(\vartheta)|}{W(\vartheta)}d\vartheta=O\left(\log^{-\alpha}\frac{1}{|h|}\right) \quad (\alpha>1),$$

then (*) holds almost everywhere. II. At almost all zeros of w one has $n^{-1}\sum_0^{n-1} p_k^2(x)\to\infty$. *W. W. Rogosinski.*

Freud, Géza. On orthogonal polynomials. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **5**, 21–27 (1955). (Hungarian)

Hungarian version of the paper reviewed above.

Tandori, Károly. Über die Cesàrosche Summierbarkeit der orthogonalen Polynomreihen. II. Acta Math. Acad. Sci. Hungar. **5**, 237–253 (1954). (Russian summary)

The author continues his investigations into the summability of Fourier series with respect to an orthonormal system of polynomials [same Acta **3**, 73–82 (1952); MR **14**, 467]. Let $\alpha(x)$ increase in $\langle a, b\rangle$ and let the $p_n(x)$ be the orthonormal polynomials with respect to $d\alpha(x)$. The main result is a generalization of a familiar theorem by Hardy and Littlewood for trigonometrical Fourier series. The main assumptions are: Let $a\leqq c\leqq c_1<d_1\leqq d\leqq b$ and let (i) the $p_n(x)$ be uniformly bounded in $\langle c, d\rangle$; (ii) $f(x)\in L^2$ in $\langle a, c_1\rangle$ and $\langle d_1, b\rangle$, and $f(x)\in L^p$, $1<p\leqq 2$, in $\langle c_1, d_1\rangle$ (with respect to $d\alpha$). If $\sigma>0$, then

$$(*) \qquad (n+1)^{-1}\sum_{k=0}^n |s_k(f, x)-f(x)|^\sigma\to 0$$

for all $x\in(c, d)$ where $\int_0^h |f(x\pm v)-f(x)|^p d\alpha(x\pm v)=o(h)$. In particular, (*) holds for α-almost all $x\in(c, d)$ for which $0<\alpha'(x)<\infty$ [$s_k(f, x)$ denotes the kth partial sum of the Fourier series of f at x]. If $\alpha'(x)>0$ p.p. in $\langle c, d\rangle$, then (*) holds p.p. in $\langle c, d\rangle$. Also if $\alpha(x)\in$ Lip, in $\langle c_1, d_1\rangle$, then (*) holds whenever f is continuous at $x\in(c_1, d_1)$. There are similar results concerning strong summability (C, r), $r>0$.

W. W. Rogosinski (Newcastle-upon-Tyne).

Special Functions

Robbins, Herbert. A remark on Stirling's formula. Amer. Math. Monthly **62**, 26–29 (1955).

It is shown by elementary methods that, in

$$n!=(2\pi)^{\frac12}n^{n+\frac12}e^{-n}\cdot e^{r_n} \quad (n=1, 2, \cdots),$$

$$\frac{1}{12n+1}<r_n<\frac{1}{12n}.$$

The proof is based on the inequalities

$$\frac{1}{12}\left(\frac{1}{p}-\frac{1}{p+1}\right)>\frac{2p+1}{2}\log\left(\frac{p+1}{p}\right)-1$$

$$>\frac{1}{12}\left(\frac{1}{p+\frac{1}{12}}-\frac{1}{p+1+\frac{1}{12}}\right).$$

S. C. van Veen (Delft).

Berger, E. R. Eine Verbesserung der Stirlingschen Formel. Z. Angew. Math. Mech. **35**, 69–70 (1955).

Approximating $e^{1/12x}$ by $(1+1/6x)^{1/2}$, the author replaces the factor $(2\pi x)^{1/2}$ in the standard Stirling approximation by $[2\pi(x+1/6)]^{1/2}$. The errors made in using these approximations are compared. A discussion of similar sort is made for $\Gamma(n+1+\theta)$ when $0\leqq\theta\leqq 1$. In particular, it is shown that

$$\Gamma(x+1)\sim(1+x)^x(1+6x/7)^{1/2}/(52/7)^{1/2}$$

when $0\leqq x\leqq 1$. *N. D. Kazarinoff* (Lafayette, Ind.).

Watanabe, Yoshikatsu, and Nakamura, Mikio. On the modified cosine functions. J. Gakugei. Tokushima Univ. Math. **5**, 39–48 (1954).

Except for a few values of the parameter n, the authors obtain the general solution of the differential equation

$$(*) \qquad \frac{d^2y}{dz^2}-\frac{2n}{z}\frac{dy}{dz}+\left(1+\frac{2n}{z^2}\right)y=0.$$

For n a positive integer or zero, a solution of (*) can be written in the form

$$V_n(z)=\left[\frac{d^n}{d\zeta^n}(\cos z(1+2\zeta)^{1/2})\right]_{\zeta=0}.$$

Many properties of the $V_n(x)$ are deduced.

F. G. Dressel (Durham, N. C.).

Džrbašyan, M. M. On the asymptotic behavior of a function of Mittag-Leffler type. Akad. Nauk Armyan. SSR. Dokl. **19**, 65–72 (1954). (Russian. Armenian summary)

For the function

$$E_{\frac12}(z, \mu)=\sum_{n=0}^\infty \frac{z^n}{\Gamma(\mu+2n)} \quad (\mu>0),$$

the author establishes the asymptotic formulas

$$E_{\frac12}(z, \mu)=\tfrac12 z^{\frac12(1-\mu)}\{\exp(z^{\frac12})+\exp[\pm i\pi(1-\mu)-z^{\frac12}]\}+O(1/z),$$

where the lower and upper signs apply respectively to $0 \leqq \arg z \leqq \pi$, $-\pi \leqq \arg z \leqq 0$; and (as $x \to +\infty$)

$$E_1(-x;\mu) = x^{\frac{1}{2}(1-\mu)} \cos[x^{\frac{1}{2}} + \tfrac{1}{4}\pi(1-\mu)] + O(1/x).$$

R. P. Boas, Jr. (Evanston, Ill.).

Carlitz, L. Note on the multiplication formulas for the Jacobi elliptic functions. Pacific J. Math. **5**, 169–176 (1955).

Let t be an odd integer and set $z = \operatorname{sn}^2 x$, of module k, $u = k^2$, $t' = \frac{1}{2}(t^2-1)$; then

$$(*) \qquad \frac{\operatorname{sn} tx}{\operatorname{sn} x} = \frac{G_1^{(t)}(z)}{G_0^{(t)}(z)} = \sum_{m=0}^{\infty} t\beta_{2m}(t) \frac{x^{2m}}{(2m)!} \quad (\beta_{2m+1}(t) = 0),$$

where $G_0^{(t)}$ and $G_1^{(t)}$ are polynomials of degree t' in z whose coefficients are polynomials with rational, integral coefficients in u and $G_0^{(t)}(0) = 1$, $G_1^{(t)}(0) = t$. From the classical [see R. Fricke, Die elliptischen Funktionen und ihre Anwendungen, v. 2, Teubner, Leipzig-Berlin, 1922, p. 197] first equality in (*) it follows that $t\beta_{2m}(t)$ is a polynomial in u, with integral coefficients for all m and all odd t. The author shows that

$$(1) \qquad \beta_{2m}(t) = H_m(t) - \sum_{p-1|2m,\, p|t} p^{-1} A_p^{2m/(p-1)}(u),$$

where $H_m(t)$ is a polynomial in u with integral coefficients, while A_p is defined by $\operatorname{sn} x = \sum_{m=0}^{\infty} A_{2m+1}(u) x^{2m+1}/(2m+1)!$. Setting also $x/\operatorname{sn} x = \sum_{m=0}^{\infty} \beta_{2m} x^{2m}/(2m)!$ and $\beta_{2m} = c/d$, $(c, d) = 1$, from (1) follows: If $d|t$, then $\beta_{2m}(t) = H_{2m} + \beta_{2m}$; if $(t, d) = 1$, then $\beta_{2m}(t)$ has integral coefficients. If $(t_1, t_2) = 1$, $2 \nmid t_1 t_2$, then $\beta_{2m}(t_1 t_2) = H_{2m} + \beta_{2m}(t_1) + \beta_{2m}(t_2)$. If p is an odd prime, $r \geqq 1$, then $\beta_{2m}(p^r) = H_{2m} + \beta_{2m}(p)$ and also $\beta_{2m}(t) = H_{2m} + \sum_{p|t} \beta_{2m}(p)$. For any integer a, $a(a^m - 1)\beta_{2m}(t)$ has integral coefficients. Defining C_p and $A_p^{(q)}$ by

$$\operatorname{sn} tx = \sum_{r=0}^{\infty} C_{2r+1} \operatorname{sn}^{2r+1} x, \quad \operatorname{sn}^{2r} x = \sum_{m=0}^{\infty} A_{2m}^{(2r)} \frac{x^{2m}}{(2m)!},$$

it follows that C_p and $A_p^{(q)}$ are polynomials in u, with integral coefficients and $\beta_{2m}(t) = t^{-1} \sum_{r=0}^{m} A_{2m}^{(2r)} C_{2r+1}$. Also, $(2m+1)C_{2m+1} \equiv 0 \pmod t$ and $pC_p/t \equiv 1 \pmod p$, for any $p|t$ hold. If $p^e|t$, $p^{e+1} \nmid t$, then $C_p \equiv tp^{-1} \pmod{p^e}$. Furthermore, if $p^{e-1}(p-1)|b$, then

$$t \sum_{s=0}^{r} (-1)^{r-s} \binom{r}{s} A_p^{(r-s)b/(p-1)} \beta_{2m+sb}(t) \equiv 0 \pmod{(p^{2m}, p^{er})}.$$

Also $\beta_{2m+b,i} \equiv (-1)^{b/2} \beta_{2m,i} \pmod{(p^{2m}, p^{e-j})}$ hold for every i, satisfying $p^{j-1} \leqq i < p^j$, where $\beta_{2m,i}$ is defined by

$$\beta_{2m}(t) = \sum_i \beta_{2m,i} u^i.$$

(1) is obtained, by combining (*) with previous results of the author [Duke Math. J. **20**, 1–12 (1953); MR **14**, 621]; the other relations follow from (1) and from each other, using again the quoted previous results of the author.

E. Grosswald (Philadelphia, Pa.).

Jordan, Henri. Eine Bemerkung über die Monotonie von sn (tK). Arch. Math. **6**, 185–187 (1955).

Let k stand for the module of the Jacobi elliptic function $z = \operatorname{sn}(tK, k)$, where $4K$ is the period and t a constant, $0 < t < 1$. Then z is monotonically increasing when k increases from 0 to 1. The proof uses the original definition of the elliptic sine and implicit differentiation. For its completion the following lemma is needed and proven: Let $p > 0$, $q > 0$; let $g(x)$ stand for a continuous function, positive for $x_0 < x < x_2$ and satisfying $p\int_{x_0}^{x_1} g(x)dx = q\int_{x_1}^{x_2} g(x)dx$ and let $h(x)$ be a positive and monotonically increasing function in the same interval. Then $p\int_{x_0}^{x_1} h(x)g(x)dx < q\int_{x_1}^{x_2} h(x)g(x)dx$.

E. Grosswald (Philadelphia. Pa.).

Robin, L., et Barret, H. Développements asymptotiques des fonctions de Bessel, lorsque leur indice augmente indéfiniment, la variable restant fixe. Centre National d'Etudes des Télécommunications, Paris, Rapp. Tech. no. **1183**, ii+11 pp. (1955).

Four or five terms of the series of the type indicated in the title are computed for $J_\nu(z)$, $I_\nu(z)$, $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$, $Y_n(z)$, and $K_n(z)$, ν complex and n a positive integer. The authors remark, "Les traités classiques . . . ne s'intéressent pas au cas où l'indice seul augmente indéfiniment." Using an algorithm given by J. Horn [Math. Ann. **52**, 340–362 (1899), p. 359] one can quickly compute as many terms of such series as one would like. *N. D. Kazarinoff.*

Mohr, Ernst. Die Maxwellsche Erzeugung der Kugelfunktionen. Math. Nachr. **12**, 273–282 (1954).

According to a theorem of Maxwell, any homogeneous solid spherical harmonic of degree n is, except for a constant multiplier, equal to r^{2n+1} times the nth derivative of r^{-1} with respect to certain directions in space of total number n, in which r is the distance from the origin. Except for trivial changes (reversing directions) Maxwell's multipole representation is unique. The author gives a new proof of this, using only analysis in the real domain and applying a minimum principle in connection with rotation of coordinates. *C. J. Bouwkamp* (Eindhoven).

Dekanosidze, E. N. Some properties of Lommel functions of two variables. Vyčisl. Mat. Vyčisl. Tehn. **2**, 97–107 (1955). (Russian)

The functions of the title are defined by the relations

$$U_\nu(w, z) = \sum_0^{\infty} (-1)^m (w/z)^{\nu+2m} J_{\nu+2m}(z),$$

$$V_\nu(w, z) = U_{-\nu+2}(w, z) + \cos(w/2 + z^2/2w + \nu\pi/2).$$

With $\xi = z^2/2w$ and $\eta = w/2$, they are solutions of the hyperbolic equation $\partial^2 U/\partial\xi\partial\eta + U = 0$. Let $\bar{D}$ be a bounded region of the $\xi\eta$-plane not containing the line $\eta = 0$, but including the line $\xi = \eta$. In solving the Cauchy problem for $\bar{D}$—with boundary conditions at $\xi = \eta$ given to be $U(\xi, \eta) = U(2\xi, 2\xi)$, $\partial U(\xi, \eta)/\partial\xi = -U_{\nu+1}(2\xi, 2\xi)$, $\partial U(\xi, \eta)/\partial\eta = U_{\nu-1}(2\xi, 2\xi)$—about fifteen complicated relations involving these functions are obtained. Of these a simpler one is

$$U_\nu(w, z) = U_\nu(z^2/w, z) + \frac{1}{2}\int_{z^2/w}^{w} J_{\nu-1}(x) J_0[\{(x - z^2/w)(x - w)\}^{1/2}]dx.$$

Several integral representations are obtained with the help of Laplace transforms; for example,

$$U_\nu(w, z) = \frac{w}{2}\int_0^1 U_{\nu-1}(wz, 0) J_0\{z(1-t)^{1/2}\} dt, \quad \nu > 0.$$

Expansions of Lommel functions in series of other Lommel functions of two variables are derived, special cases of which were previously known. An asymptotic expansion in z of $U_\nu(cz, z)$ is given. Many of these results should be useful for computational purposes. Because of the complexity of the formulas involved, few of the details of derivation of results are included. *N. D. Kazarinoff* (Lafayette, Ind.).

Lammel, Ernesto. On the solutions of the differential equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u(x, y, z)=0$$

which possess axial symmetry. Univ. Nac. Tucumán. Rev. Ser. A. 10, 27–73 (1954). (Spanish. German summary)

In the space of solutions of the three-dimensional potential equation which are symmetric with respect to the z-axis, the author determines first the unique homogeneous polynomial Φ of degree n in z and ρ, where $\rho^2=x^2+y^2$, for which the term in z^n has unit coefficient. The corresponding stream function Ψ_n is determined, after which the series

$$A_0+\sum_{\nu=1}^{\infty}A_\nu\Phi_\nu, \quad \sum_{\nu=1}^{\infty}A_\nu\Psi_\nu$$

are discussed. *E. F. Beckenbach* (Los Angeles, Calif.).

Differential Equations

Have

***Coddington, Earl A., and Levinson, Norman. Theory of ordinary differential equations.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955. xii +429 pp. $8.50.

This excellent book contains many topics which at present are of interest, such as asymptotic behavior of linear and nonlinear systems, boundary-value problems, both on finite and infinite intervals, stability, questions of perturbations, and the Poincaré-Bendixson theory of two-dimensional autonomous systems. Besides classical theorems there are to be found many results published during the last ten years, particularly results of the authors. In addition, new results of the authors are presented. The presentation, characterized by the exclusive use of vector and matrix notations, is elegant and lucid, even if brief.

The first two chapters deal with the single equation $x'=f(t, x)$ and with the system written in the same form, where x denotes a vector and f a vector function. The authors have succeeded in sixty pages not only in introducing all fundamental notions and in deriving the Cauchy-Peano and Carathéodory existence theorem, the method of successive approximations and theorems on continuation and dependence on initial conditions and parameters in the real as well as complex domain, but also in giving general sufficient conditions for the uniqueness of the solutions and for the convergence of successive approximations.

Chapters 3 through 12 are on linear equations. Chap. 3 contains linear homogeneous and nonhomogeneous systems with the variation-of-constants formula in matrix form, linear systems with constant and periodic coefficients, linear equations of an arbitrary order, and a theorem of the asymptotic behavior of the system $x'=(A+V(t)+R(t))x$, where A is a constant matrix and

$$\int_0^\infty |R(t)|\,dt<\infty, \quad \int_0^\infty |V'(t)|\,dt<\infty$$

and $\lim_{t\to\infty} V(t)=0$. Chap. 4 and 5 are concerned with linear systems in the complex domain with isolated singularities: $w'=(z-z_0)^{-\mu-1}\tilde{A}(z)$ where $\mu\geqq 0$ is an integer, $\tilde{A}(z)$ is analytic for $|z-z_0|<a$, $a>0$, and $\tilde{A}(z_0)\neq 0$. According as $\mu=0$ or $\mu\geqq 1$, the point z_0 is called a singular point of the first kind, or a singular point of the second kind. A novel treatment of the singular point of the first kind for systems is given, several classical results on the equation of nth order are introduced and the asymptotic expansions of the solutions in the neighborhood of a singular point of the second kind lying at $z=\infty$ are investigated. In the following chapter the system $x'=\rho^\mu A(t, \rho)x$ is considered, where $a\leqq t\leqq b$, ρ is a large parameter, $\mu\geqq 1$ is an integer and $A(t, \rho)=\sum_{k=0}^{\infty}\rho^{-k}A_k(t)$ with $A_k(t)$ continuous for $a\leqq t\leqq b$. In the case that the matrix $A_0(t)$ has distinct characteristic roots for $t\in[a, b]$ asymptotic expansions of the solutions with respect to ρ are given. The transition points are not considered.

Chapters 7 through 12 are on boundary-value problems. Chap. 7 deals with regular self-adjoint eigenvalue problems. The Green's function $G(t, \tau)$ is introduced and by investigating the linear integral operator $\mathcal{G}f(t)=\int_a^b G(t, \tau)f(\tau)d\tau$ the existence of eigenvalues and the expansion and completeness theorems are proved. Chap. 8 contains comparison theorems for $(p(t)x')'+g(t)x=0$, the oscillation theorem, and a theorem on eigenvalues and eigenfunctions of the eigenvalue-problem with periodic boundary conditions for $(p(t)x')'+(\lambda\mu(t)-q(t))x=0$. Chap. 9 is concerned with singular self-adjoint boundary-value problems for second-order equations. The notion of the limit-point and limit-circle cases at infinity is explained, a simplified proof of the completeness and expansion theorems in the limit-point case is given, and then the limit-circle case examined. In the following chapter the theory of chap. 9 is extended to the nth-order case. In both chapters the knowledge of the Lebesgue and Lebesgue-Stieltjes integral is needed. Chap. 11 treats the algebraic properties of linear boundary-value problems on a finite interval, and chap. 12 the non-self-adjoint boundary-value problems by means of the Cauchy integral method.

Chap. 13 contains theorems on asymptotic, orbital and conditional stability and on asymptotic behavior of the solutions of certain nonlinear systems. The Lyapunov theory of stability is not mentioned. Chap. 14 deals with the Poincaré method of perturbations. First the systems $x'=f(t, x, \mu)$ and $x'=f(x, \mu)$, where μ is a small parameter and f has period 2π in t, are considered in the case when certain Jacobians do not vanish, then the quasilinear systems $x'=Ax+\mu f(t, x, \mu)$ and $x'=Ax+\mu f(x, \mu)$ are treated, where A is a constant matrix with at least one characteristic root of the form iN (N an integer). In the following chapter the two-dimensional linear system $x_1'=ax_1+bx_2$, $x_2'=cx_1+dx_2$ $(ad-bc\neq 0)$ is discussed and then the perturbation theory is given for the system $x_1'=ax_1+bx_2+f_1(x_1, x_2)$, $x_2'=cx_1+dx_2+f_2(x_1, x_2)$, where $f_1=o(r)$ and $f_2=o(r)$ as $r=(x_1^2+x_2^2)^{1/2}\to 0$. Chap. 10 is concerned with the Poincaré-Bendixson theory of two-dimensional autonomous systems and the last chapter is devoted to differential equations on a torus.

To every chapter there are added plenty of problems accompanied by hints for solution which, in many cases, give additional material not considered in the text. The authors do not mention the historical origins of the theory nor do they remark where they present new results. However, at the end of the book there are given references to every chapter.

The elegant manner of presentation and the choice of material, most of which is not to be found in standard textbooks on differential equations, make the reading of the book a pleasure. *M. Zlámal* (Brno).

Pliś, A. On the problem of non-local existence for first integrals of a system of ordinary differential equations. Bull. Acad. Polon. Sci. Cl. III. 3, 63–67 (1955).

An example is given of a system (S) $x_i' = f_i(x, y_j)$ $(i, j = 1, 2)$ with $f_i \in C^\infty$ in E_3 such that if $z(x, y_j)$ is continuous in E_3 and is constant on the integrals of (S) then z is constant throughout E_3. (S) is constructed in such a way that its integral of ramification (in the sense of Ważewski) is dense in E_3. *F. A. Ficken* (Knoxville, Tenn.).

Keil, Karl-August. Das qualitative Verhalten der Integralkurven einer gewöhnlichen Differentialgleichung erster Ordnung in der Umgebung eines singulären Punktes. Jber. Deutsch. Math. Verein. 57, 111–132 (1955).

Consider the real differential system

$$\dot{x}(t) = ax + by + f(x, y), \quad \dot{y}(t) = cx + dy + g(x, y),$$

where $f, g \in C^{(1)}$ near the origin and there contain only higher-order terms. For the degenerate case $ad - bc = 0$ but $a^2 + b^2 + c^2 + d^2 > 0$, the author analyses the quantitative behavior of the solution curves near the origin, which is assumed to be an isolated critical point. *L. Markus.*

Katō, Tizuko. Sur les points singuliers des équations différentielles ordinaires du premier ordre. III. Nat. Sci. Rep. Ochanomizu Univ. 5, 1–4 (1954).

[For parts I–II see same Rep. 2, 13–17 (1951); 4, 36–39 (1953); MR 14, 274; 15, 126.] Let $t_0 < 0$ be large in magnitude, let $P(x, y)$, $Q(x, y)$ be polynomials and let $Q(x, 0) \neq 0$. If the equation $y dy/dt = P(e^t, y)/Q(e^t, y)$, has a solution $y(t)$ such that $y(t_0 - m\omega) = 0$ for $m = 0, 1, 2, \cdots$ and some $\omega > 0$ and, if $y dy/dt = P(0, y)/Q(0, y)$ has a periodic solution $\phi(t, t_0)$ of real period ω such that $\phi(t_0, t_0) = 0$, then in a certain domain of the complex t-plane

$$y(t) = \phi(t, t_0)\left[1 + \sum_{j=1}^{\infty} p_j(t, t_0) e^{jt}\right],$$

where the functions p_j have period ω and the points $t_0 - m\omega$ are algebraic critical points of the p_j. *N. Levinson.*

Hartman, Philip, and Wintner, Aurel. On linear, second order differential equations in the unit circle. Trans. Amer. Math. Soc. 78, 492–500 (1955).

The differential equation (*) $w''(z) + p(z)w(z) = 0$ is said to be disconjugate on a set S if no solution of (*) has more than one zero for $z \in S$. In the present paper the authors re-prove and extend some criteria for the disconjugacy of (*) on $|z| < 1$ given earlier by the reviewer [Bull. Amer. Math. Soc. 55, 545–551 (1949); Amer. J. Math. 76, 689–697 (1954); MR 10, 696; 16, 131]. *Z. Nehari.*

Latyševa, K. Ya. On solution in closed form of linear differential equations with polynomial coefficients. Dokl. Akad. Nauk SSSR (N.S.) 101, 405–408 (1955). (Russian)

The author examines the linear differential equation

$$\sum_{i=0}^{n} P_i(x) d^{n-i}y/dx^{n-i} = 0$$

with polynomial coefficients $P_i(x) = \sum_{j=\pi_i}^{\sigma_i} p_{ij} x^j$ and proves the possibility of the existence of solutions of the closed form $y = e^{Q(x)} S(x)$, where in the case that the rank $p = 1 + \max_{1 \leq i \leq n} (\sigma_i - \sigma_0)/i$ and the antirank

$$m = -1 - \min_{1 \leq i \leq n} (\pi_i - \pi_0)/i$$

are integers the function $S(x)$ is rational and $Q(x)$ has the form

$$1)\ \sum_{s=1}^{p} \tau_{p-s} x^s/s, \qquad 2)\ \sum_{s=1}^{m} \lambda_{m-s}/s x^s,$$

$$3)\ \sum_{s=1}^{p} \tau_{p-s} x^s/s + \sum_{s=1}^{m} \lambda_{m-s}/s x^s,\ 4)\ 0$$

according as 1) $p > 0$, $m \leq 0$, 2) $p \leq 0$, $m > 0$, 3) $p > 0$, $m > 0$, 4) $p \leq 0$, $m \leq 0$. *M. Zlámal* (Brno).

McKelvey, Robert W. The solutions of second order linear ordinary differential equations about a turning point of order two. Trans. Amer. Math. Soc. 79, 103–123 (1955).

The asymptotic behavior for large $|\lambda|$ of solutions of the equation

$$(*) \qquad d^2u/dx^2 - [\lambda^2 p_0(x) + \lambda p_1(x) + Q(x, \lambda)]u = 0,$$

$$Q(x, \lambda) = \sum_{i=0}^{\infty} q_i(x)/\lambda^i,$$

is investigated over a bounded interval of the real axis upon which $p_0(x)$ has a single zero which is of the second order. The functions $p_i(x)$ and $q_i(x)$ are assumed to be analytic on the interval. The discussion consists of two parts. In the first a "related equation" is constructed whose coefficients are identical with those of (*) up to terms of order λ^{-n}, n an arbitrary positive integer. In the second it is rigorously demonstrated that solutions of this related equation asymptotically represent those of (*). These representations depend, of course, upon arg (λ), while different choices of the approximating solutions are made near the turning point and on intervals of finite distance from it.

Langer previously has given first approximations to solutions of (*) by means of Whittaker functions [Trans. Amer. Math. Soc. 36, 90–106 (1934)]. Evans has found complete asymptotic series for the solutions [Thesis, Univ. of Minnesota, 1951]. The author's algorithm for obtaining complete asymptotic series for solutions of (*) is simpler than that of Evans in that it involves only single quadratures of elementary functions of $p_i(x)$ and $q_i(x)$. It also avoids an awkward normalization of (*) which Langer employed.

Assume the zero of $p_0(x)$ to lie at $x = 0$, let $W(\xi)$ denote any solution of the specialized Whittaker equation

$$d^2w/d\xi^2 - [1/4 - K(\lambda)/\xi - 3/16\xi^2]w = 0,$$

and let $\xi = 2\lambda \int_0^x p_0^{1/2}(s)\, ds$. The differential equation satisfied by the functions $v(x, \lambda) = p_0^{-1/4}(x) W[\xi(x)]$ is used as a first approximation to (*). Paralleling Langer's discussion [Trans. Amer. Math. Soc. 67, 461–490 (1949); MR 11, 438], a sequence of equations is found of which successive elements improve in their approximation to (*). Their solutions have the form $E_0(x, \lambda)v(x, \lambda) + E_1(x, \lambda)v'(x, \lambda)/\lambda$. The coefficients of powers of λ^{-1} in E_0 and E_1 are determined in an iterative fashion. The crucial feature of the discussion is the determination of the quantity $K(\lambda)$ as a polynomial in λ^{-1} so as to remove singularities which otherwise would have been present in E_0 and E_1 as obtained by modifying Langer's 1949 discussion to fit (*). [Reviewer's remark: The statement immediately preceding relation (7.6) may be improved to read $O(\lambda^{-n-2}) \int_{\xi(x)}^{\xi(b)} \xi^{-1} d\xi = O(\lambda^{-n-2} \ln \lambda)$. This causes a decrease in magnitude of some of the error terms in Theorems 1 and 3 by the factor $\lambda^{-1} \ln \lambda$.] *N. D. Kasarinoff.*

Marić, V. On the asymptotic behaviour of integrals of a class of nonlinear differential equations of second order. Srpska Akad. Nauka. Zb. Rad. **43**. Mat. Inst. **4**, 27–40 (1955). (Serbo-Croatian. English summary)

This paper extends to a new class of differential equations work of Avakumović [Acad. Serbe Sci. Publ. Inst. Math. **1**, 101–113 (1947); MR **10**, 455]. Consider $\rho(x)=x^\alpha L(x)$, where $L(x)$ satisfies the conditions $L(xt)/L(x)\to 1$, as $x\to\infty$, for all $t>0$ and $L(x)=c(x)\exp\int_a^x \epsilon(t)t^{-1}dt$, in which $c(x)\to c$, $\epsilon(x)\to 0$, as $x\to\infty$. Let $L_n(x)=\prod_1^n(\log_\nu x)^{\xi_\nu}$, where the ξ_ν are arbitrary constants and "$\log_\nu$" indicates the νth iteration of "log." The equation investigated is of the form $y''(x)=\rho_k(x)y^\lambda L_n[1/y(x)]$, in which $\lambda>1$, $\alpha>0$, $k>1$, and $\rho_k(x)=\rho(\exp x^{1/k})$. The asymptotic behavior of the unique solution of this equation satisfying $y(0)=1$, $y(x)\to 0$, as $x\to\infty$ is found. The solution $y(x)$ behaves essentially as $[x^\beta\rho_k(x)L_{n-1}(x)]^{1/(1-\lambda)}$, β a known constant.

N. D. Kazarinoff (Lafayette, Ind.)

Biernacki, Mieczysław. Sur le nombre minimum des zéros des intégrales de l'équation $y^{(n)}+A(x)y=0$. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **7** (1953), 15–18 (1954). (Polish and Russian summaries)

The author proves the following theorem. Let

$$y^{(n)}+A(x)y=0,$$

where $A(x)$ is a positive continuous function for all real x. If n is even, there exists a (non-trivial) solution with at least n real zeros; if n is odd, there exists a (non-trivial) solution with at least $(n-1)$ real zeros.

The author conjectures that the functions

$$A(x)=k(1+x^2)^{-n},$$

for some small $k>0$, show that the results of the theorem are the best possible. *L. Markus* (Princeton, N. J.).

✶Bulgakov, B. V. Kolebaniya. [Oscillations.] Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 891 pp. (1 plate). 29.40 rubles.

This very large volume, nearly 900 pages, is a posthumous work by the man who for a number of years occupied the chair of mathematics and mechanics of the University of Moscow. The first two of the three parts which compose it, appeared as a first volume of this contemplated work in 1949 [MR **12**, 335]. The third part, left incompleted at the author's death in 1952, was prepared for publication by Ya. N. Roĭtenberg.

The first part (first three chapters) is strictly preparatory and covers material amply expounded in the literature: matrices (97 pages); operator calculus (55 pages); equations of dynamics (33 pages). The subject of oscillations properly begins with the fourth chapter (p. 203). The book is destined above all for applied mathematicians—it will give only moderate satisfaction to others. It is replete with engineering problems, very well and very fully discussed. Part II deals with systems of one degree of freedom and Part III with systems of a finite number of degrees of freedom. Not content to plant directly before the reader the appropriate differential equations of the oscillations, the author always indicates their derivation and generally starts from energetic considerations. The mathematical treatment is always dealt with carefully and with due respect.

A brief description of the various chapters from the fourth on (first of Part II) follows.

Part II. Systems of one degree of freedom. Ch. 4. Equations of motion, canonical and linear systems. Linear systems with constant coefficients; oscillatory systems as linear filters; automatic regulation; general solution; stability of oscillations. Linear systems with a lag (difference-differential equations with constant coefficients). Ch. 5. Self-oscillations of non-linear systems. Conservative systems. Dissipative systems. Self-oscillatory systems: clocks, vacuum tubes, steam engine. Method of small parameters of Poincaré. Quasi-linear systems, their approximations. Application to vacuum tubes and steam engines. Averaging method. Ch. 6. Forced oscillations. Approximations. Stationary systems with same period as forced oscillation. Systems with non-linear forced oscillations, also with viscous friction.

Part III. Systems of a finite number of degrees of freedom. Ch. 7. Small oscillations around a given state. Variation equations. Reduction to the first order. Stability. Influence of variable parameters. Application to steam engine with regulator. Small oscillations of a natural system around a steady state. Ch. 8. Linear system with constant coefficients, their oscillations, self-oscillations and stability. Ch. 9. Passive systems (i.e. without exterior source of energy). Linear conservative systems with constant coefficients. Synthesis of bipolar and quadripolar systems. Ch. 10. Linear regulated systems, with application to gyroscopic stabilization and to ships with automatically regulated rudder. Nyquist criterium. Ch. 11. Linear systems with periodic coefficients. Ch. 12. Nonlinear systems. Method of Poincaré (small coefficients). Ch. 13. Averaging method. Appendix: On motions with forced oscillations of high frequency [translation of Compositio Math. **7**, 390–427 (1940); MR **1**, 236]. The volume concludes with a rather complete index and an ample bibliography. *S. Lefschetz.*

Krasovskiĭ, N. N. On inversion of K. P. Persidskiĭ's theorem on uniform stability. Prikl. Mat. Meh. **19**, 273–278 (1955). (Russian)

This note deals with the uniform stability of the origin for a system

$$\dot{x}=X(x;t),\quad X(0;t)=0,\quad \|x\|\leqq H,\quad t>0,$$

where x, X are n-vectors and the modulus is $\sup\|x_i\|$. The uniform stability is in the sense of Persidskiĭ (with respect to the time). In substance the author proves: A necessary and sufficient condition for the uniform stability of the origin is the existence of a Lyapunov function $v(x;t)$ which is definite positive, is small uniformly in t ($t\geqq t_0>0$), and has a time derivative $\dot{v}$ (along the trajectories) which is negative. The necessity of the condition was already proved by Persidskiĭ [Dissertation, Moscow, 1946].

S. Lefschetz (Princeton, N. J.).

Staržinskiĭ, V. M. On the stability of the trivial solution of a differential equation of second order with periodic coefficients. Inžen. Sb. **18**, 119–138 (1954). (Russian)

The author investigates stability regions for the system $\dot{x}_r=\sum_{s=1}^2 p_{rs}(t)x_s$ $(r=1,2)$, where the p_{rs} have period ω, p_{12} and p_{21} have constant signs, and $\alpha=\int_0^\omega p_{11}dt\geqq\beta=\int_0^\omega p_{22}dt$. His method is a direct extension of that of Lyapunov [C. R. Acad. Sci. Paris **123**, 1248–1252 (1896); **128**, 910–913, 1085–1088 (1899)]. For $\alpha=\beta$ the system can be reduced to the Lyapunov case, and criteria in terms of the Lyapunov series $\sum A_n$ are found; there are detailed numerical investigations for the equations $\ddot{x}+(\delta\sin t)\dot{x}+x=0$, $\ddot{x}+a\dot{x}+p(t)x=0$, $\ddot{x}+a\dot{x}+C(1+\lambda\cos t+\mu\cos 2t)x=0$. The general case $\alpha>\beta$ is briefly considered, and as an example there is the system

$$dx_1/dt=x_2,\quad dx_2/dt=-\gamma x_1-a(1+\delta\cos t)x_2.$$

The work is summarised in pp. 495–497 of the author's survey in Prikl. Mat. Meh. **18**, 469–510 (1954) [MR 16, 249]. *F. V. Atkinson* (Oxford).

Staržinskiĭ, V. M. Remark on the investigation of stability of periodic motions. Prikl. Mat. Meh. **19**, 119–120 (1955). (Russian)

For the system $\dot{x}_r=\sum_{s=1}^{2} p_{rs}(t)x_s$ $(r=1, 2)$, with

$$p_{rs}(t+\omega)\equiv p_{rs}(t),$$

the author proposes the substitution

$$y_1=x_1\cos(k\pi t/\omega)+x_2\sin(k\pi t/\omega),$$
$$y_2=-x_1\sin(k\pi t/\omega)+x_2\cos(k\pi t/\omega),$$

with a view to a reduction to the form $\ddot{z}+p(t)z=0$ with $p(t+\omega)\equiv p(t)$ and $p(t)>0$, known stability criteria being then applicable. The device succeeds if k is a sufficiently large integer, and the $p_{rs}(t)$ suitably smooth.
F. V. Atkinson (Oxford).

Tuzov, A. P. Stability questions for a certain regulating system. Vestnik Leningrad. Univ. **1955**, no. 2, 43–70 (1955). (Russian)

In several articles Erugin [Prikl. Mat. Meh. **14**, 459–512, 659–664 (1950); **16**, 620–628 (1952); MR **12**, 412; **14**, 376] investigated the stability in the whole plane of a system

$$(1)\qquad \dot{x}=a_{11}x+a_{12}y+f(x),\quad \dot{y}=a_{21}x+a_{22}y,$$

under the assumptions for $x\neq 0$:

$$(2)\qquad \begin{cases}(a_{11}+a_{22})x^2+xf(x)<0,\\ (a_{11}a_{22}-a_{21}a_{12})x^2+a_{22}xf(x)>0.\end{cases}$$

The present paper deals with the same sort of question for a system

$$(3)\qquad \begin{cases}\dot{x}=a_{11}x+a_{12}y+a_{13}z+f(x),\\ \dot{y}=a_{21}x+a_{22}y+a_{23}z,\\ \dot{z}=a_{31}x+a_{32}y+a_{33}z,\end{cases}$$

where $f(x)$ satisfies conditions of similar type to (2). The author arrives at his highly complicated results (local asymptotic stability, stability in the large, stability as a whole) by a tireless consideration of particular cases, imposing also very often highly special conditions. One of the simplest is for example $\lambda x^2<xf(x)<\mu x^2$. It would be hopeless to give any sort of details. *S. Lefschetz.*

Skalkina, M. A. On a connection between stability of solutions of differential and finite-difference equations. Prikl. Mat. Meh. **19**, 287–294 (1955). (Russian)

Consider the system

$$(1)\qquad \dot{x}=X(t;x),$$

where x, X are n-vectors, $X(t;0)=0$, and in relation to (1) the finite-difference system

$$(2)\qquad x^h_{m+1}=x_m^h+hX(t_0+m^h:x_m^h),$$

where x_m^h is still an n-vector: $x_m^h=x^h(t_0+mh)$, $x(t_0)=x_0$ in a certain closed region G and $x^h(t_0)=x_0^h$ in G. What are the stability relations between (1) and (2), at least for h sufficiently small? These are discussed by the author for constant coefficients, stationary or non-stationary periodic systems, non-stationary systems. The basic propositions are the following. I. If (1) is a system with constant coefficients a.s. (=asymptotically stable) at the origin, so is (2) for h sufficiently small. In fact, if r_j are the characteristic numbers of (1), then it is sufficient to have

$$0<h<\inf\left|-\frac{2\operatorname{Re}r_j}{|r_j^2|}\right|.$$

II. If (1) is stable in the first approximation, so is (2) for h sufficiently small. III. Let (1) be periodic or stationary periodic and a.s. Corresponding to any small ϵ there exists a $\delta(\epsilon)$ and $h^0(\epsilon)$ such that any solution of (2) with $h<h^0$ such that $\|x^h(t_0)\|<\delta$ satisfies $\|x^h(t_0+mh)\|<\epsilon$ whatever the positive integer m. IV. If (1) is a.s. uniformly for $t_0>0$ and $x_0\in D$, then given any ϵ there exist $h_0(\epsilon)$ and $\eta(\epsilon)$ such that for every solution of (2) corresponding to $h<h_0$ and any m we have $\|x^h(t_0+mh)\|<\epsilon$ if $\|x_0^h\|<\eta$. (The norm taken throughout is sup $|x_h|$ for a vector $(x_1,\cdots,x_n)$.)
S. Lefschetz (Princeton, N. J.).

Bottema, Oene. On the stability of the equilibrium of a linear mechanical system. Z. Angew. Math. Phys. **6**, 97–104 (1955).

The differential equations of a linear mechanical system of two degrees of freedom are investigated for stable equilibrium. The author is particularly interested in the case in which unstable equilibrium approaches stability as the "damping" forces tend to zero. *E. Pinney.*

Lillo, James C., and Seifert, George. On conditions for stability of solutions of pendulum-type equations. Z. Angew. Math. Phys. **6**, 239–243 (1955).

The authors consider the equation

$$y(\theta)y'(\theta)=g(\theta)-\alpha f(\theta)y(\theta),$$

where α is a positive parameter, $f(\theta)$ and $g(\theta)$ are twice differentiable periodic functions with the period 2π, $f(\theta)>0$ for all θ, $g(\theta)$ has a finite number of simple zeros in a period, and the integral of $g(\theta)$ over a period is positive. It is known that the nonexistence or existence of a solution $y(\theta)$ with period 2π depends on whether or not α exceeds a critical value $\alpha_c>0$. The purpose of this note is to give some new upper and lower bounds for α_c. In several places the details of the derivations of these bounds are not entirely clear to the reviewer. *L. A. MacColl* (New York, N. Y.).

Caughey, T. K. The existence and stability of ultraharmonics and subharmonics in forced nonlinear oscillations. J. Appl. Mech. **21**, 327–335 (1954).

This is a heuristic investigation of standard type of the fundamental, third harmonic, and subharmonic of order one-third of the equation

$$\frac{d^2y}{dt^2}+K\frac{dy}{dt}+\omega^2y+\mu y^3=P\cos(\Omega t+\alpha),\quad \mu\text{ small.}$$

E. Pinney (Berkeley, Calif.).

Bulgakov, N. G. Oscillations of quasilinear autonomous systems with many degrees of freedom and a nonanalytic characteristic of nonlinearity. Prikl. Mat. Meh. **19**, 265–272 (1955). (Russian)

The paper deals with the oscillations for a system

$$(1)\qquad \dot{x}=Ax+\mu f(x),$$

where x, f are n-vectors, A is a constant matrix, μ a small parameter. It is assumed that f is of class C^1 in a certain region D. Let A possess a single zero characteristic root and a certain number of pure complex roots $\pm p_j\lambda i$, p_j a positive integer, λ a positive real number. If m is the total multiplicity of these roots, then the linear approximation (2)

$\dot{x}=Ax$ has m linearly independent periodic solutions of period $\omega=2\pi/\lambda$, say $\varphi^{(1)}(t), \cdots, \varphi^{(m)}(t)$. The adjoint system to (2) has likewise m such solutions $\psi^{(h)}(t)$.

Let $T=\omega(1+\mu\alpha)$, $\alpha=\alpha(\mu)$, be a possible period for a periodic solution of (1) and set $\alpha(0)=\alpha^*$. The change of variables $t=\tau(1+\mu\alpha)$ replaces (1) by

$$(3)\qquad \frac{dx}{d\tau}=Ax+\mu F(x, \alpha, \mu), \quad F=(1+\mu\alpha)f+\alpha Ax.$$

The possible periodic solution of period T of (1) becomes a periodic solution of period ω of (3). Set

$$(4)\qquad x^*(\tau)=\sum M_j\varphi^{(j)}(\tau).$$

Theorem: A necessary and sufficient condition to have (4) lying in D generate a periodic solution $x(\tau)$ of period ω of (3) which for $\mu=0$ reduces to (4) are

$$\int_0^\omega F(x^*, \alpha^*, 0)\psi^{(h)}(\tau)d\tau$$

$$=\int_0^\omega f(x^*)\psi^{(h)}(\tau)d\tau+\alpha^*\omega\sum M_kA_{hk}=0, \quad h=1, 2, \cdots, m,$$

where the (constants) A_{hk} are given by

$$A_{hk}=\sum_s \psi_s^{(h)}(\tau)\frac{d\varphi_s^{(k)}(\tau)}{d\tau}.$$

S. Lefschetz (Princeton, N. J.).

Sibuya, Yasutaka. Sur un système des équations différentielles ordinaires linéaires à coefficients périodiques et contenant des paramètres. J. Fac. Sci. Univ. Tokyo. Sect. I. 7, 229–241 (1954).

The system is $\dot{X}=A(t; z_\alpha)X+V(t; z_\alpha)$ where V and X are n-component complex vectors, A is a square matrix, t is a scalar variable ($\dot{}=d/dt$), and the z_α ($\alpha=1, \cdots, r$) are complex parameters. In the region $\|z\|<\delta$, $-\infty<t<\infty$, A and V are continuous, ω-periodic in t, and analytic in the z_α. The author obtains a system $\dot{Y}=B(z_\alpha)Y+W(z_\alpha)$ by a transformation $X=P(t; z_\alpha)Y+U(t; z_\alpha)$ which is ω-periodic in t, analytic in z_α, and has nonsingular $P(t; 0)$. The homogeneous case is studied first. Sufficient conditions are given for the convergence of the formal power series used.

F. A. Ficken (Knoxville, Tenn.).

Sibuya, Yasutaka. Sur les solutions périodiques d'un système des équations différentielles ordinaires non linéaires à coefficients périodiques. J. Fac. Sci. Univ. Tokyo. Sect. I. 7, 243–254 (1954).

The system is reduced to $\dot{x}_j=\lambda_jx_j+\delta_jx_{j-1}+f_{2j}(t; x_i)$, where $i, j=1, \cdots, n$, λ_j and δ_j are constants such that $\delta_j\neq0$ implies $\lambda_{j-1}=\lambda_j$, and for $\|x\|<\eta$ the f_{2j} are convergent power series in the x_i, with no terms of degree <2, and with coefficients continuous and ω-periodic in t. Relations between the coefficients of an ω-periodic formal solution are investigated, and a rather elaborate general expression for a periodic solution is found. Convergence questions are also studied.

F. A. Ficken (Knoxville, Tenn.).

de Castro, Antonio. Sull'esistenza ed unicità delle soluzioni periodiche dell'equazione $\ddot{x}+f(x, \dot{x})\dot{x}+g(x)=0$. Boll. Un. Mat. Ital. (3) 9, 369–372 (1954).

The differential equation in the title is supposed to satisfy the following conditions: (i) $f(x, v)$ and $g(x)$ are continuous and Lipschitzian; (ii) $g(x)$ is increasing, $xg(x)>0$, for $x\neq0$, $\int_0^{\pm\infty} g(\xi)d\xi=+\infty$; (iii) $f(0, 0)<0$; (iv) $f(x, v)>0$, for $|x|>a$; (v) there exist numbers N, $\alpha>0$ such that $\int_{-a}^{a} f(\xi, v(\xi))d\xi\geqq\alpha>0$, for every $v(\xi)>N$. By means of a fixed-point argument in a suitably constructed ring-shaped domain the author proves the existence of a periodic solution, which is then shown to be unique.

W. Wasow (Los Angeles, Calif.).

Mitropol'skiĭ, Yu. A. On the effect on a nonlinear oscillator of a "sinusoidal" force with modulated frequency. Ukrain. Mat. Ž. 6, 442–447 (1954). (Russian)

This is a continuation of two earlier papers [Prikl. Mat. Meh. 14, 139–170 (1950); Inžen. Sb. 15, 89–98 (1953); MR 12, 181; 16, 822]. Here the author discusses the solutions of

$$\ddot{x}+\omega^2x=\epsilon f(x, \dot{x})+E\sin\theta, \quad \dot{\theta}=\nu(t)=\nu_0+\Delta\nu_0\cos\Omega t.$$

The methods of the first cited paper (à la Krylov-Bogolyubov) are applicable for $\Delta\nu_0$ or Ω small relative to ν_0. Two special cases are discussed mainly by graphical methods.

S. Lefschetz (Princeton, N. J.).

Huang, T. C. Harmonic oscillations of nonlinear two-degree-of-freedom systems. J. Appl. Mech. 22, 107–110 (1955).

The method of Poincaré is applied to investigate the fundamental vibration of two masses connected by a nonlinear spring when one is fastened to a foundation by a linear spring and the other is subjected to a harmonically oscillating force.

E. Pinney (Berkeley, Calif.).

Morris, G. R. A differential equation for undamped forced non-linear oscillations. I. Proc. Cambridge Philos. Soc. 51, 297–312 (1955).

Let (1) $\ddot{x}+2x^3=e(t)$, where $e(t)$ is a real continuous even function with least period 2π and let $\omega=2\pi^{-1}\int_0^1(1-u^4)^{-1/2}du$. Then the principal result of this paper is the following theorem. If m is a positive integer, (1) has an infinity of periodic solutions of least period $2m\pi$; further, for any sufficiently large integer k, there is at least one such solution with positive maxima and negative minima at which $|x|=k\omega+O(1)$ and with no other stationary points.

L. Markus (Princeton, N. J.).

Manacorda, Tristano. Studio di un circuito non lineare col metodo stroboscopico di N. Minorsky. Boll. Un. Mat. Ital. (3) 8, 281–285 (1953).

The stroboscopic method is used to investigate oscillations in a circuit consisting of inductance, resistance, and capacity, where the inductance is a nonlinear function of the current, and the capacity is an explicit periodic function of the time. The method indicates that if the constants satisfy certain natural conditions, there exists a unique stable oscillation, with a period which is twice that of the capacity.

L. A. MacColl (New York, N. Y.).

Cunningham, W. J. Simultaneous nonlinear equations of growth. Bull. Math. Biophys. 17, 101–110 (1955).

The equations are

$$dx/dt=(a_x/k_x)[k_x-x-f_x(y)]x,$$
$$dy/dt=(a_y/k_y)[k_y-y-f_y(x)]y,$$

where a_x, a_y, k_x, and k_y are constants, and where $f_x(y)$ and $f_y(x)$ are continuous, single-valued, and differentiable, and vanish with their arguments. The equations' singular points in the phase plane are identified as to position and as to stability type. Some phase-plane trajectories, calculated on

an analogue computer, are given for a case in which f_x and f_y are quadratic. *E. Pinney* (Berkeley, Calif.).

Servranckx, R. Résolution d'une équation associée à l'équation de Bessel. Acad. Roy. Belg. Bull. Cl. Sci. (5) 41, 556–559 (1955).

The equation considered is $x^2f''+xf'-(x^2+\rho^2)f=x^\nu I_\mu(x)$, with $\rho\geqq 0$, $\mu\geqq 0$, $\nu\geqq 0$. The author obtains its solutions in series of the functions $I_{\mu+\nu+2n}(x)$ $(n=0, 1, \cdots)$.
N. D. Kazarinoff (Lafayette, Ind.).

Storlazzi, Rosetta. Sul moto di un punto in un piano con assegnate condizioni per la velocità. Matematiche, Catania 9, 106–112 (1954).

This note gives a detailed discussion of the solution, by elementary methods, of the system of equations

$$\dot{x}=\lambda x \sin \mu t+\lambda y \cos \mu t,$$
$$\dot{y}=\lambda x \cos \mu t-\lambda y \sin \mu t,$$

where λ and μ are constants. *L. A. MacColl.*

Epheser, Helmut, Über die Existenz der Lösungen von Randwertaufgaben mit gewöhnlichen, nichtlinearen Differentialgleichungen zweiter Ordnung. Math. Z. 61, 435–454 (1955).

Let A denote the class of functions $y(x)$ on $a\leqq x\leqq b$ with absolutely continuous $y'(x)$. Problem P_j is to find y in A satisfying the equation

$$y'(x)-y'(a)=\int_a^x f(t, y(t), y'(t))dt$$

and boundary conditions C_j $(j=1, 2, \text{or } 3)$: 1) $y(a)=0=y(b)$; 2) $y(a)=0=y'(b)$; 3) $y'(a)=0=y(b)$. Here $f(x, u, v)$ is defined for $a\leqq x\leqq b$, $-\infty<v<\infty$, and either $-\infty<u<\infty$ or $y_1(x)<u<y_2(x)$ with y_1 and y_2 in A. f is to be continuous in (u, v) with x fixed, and measurable in x with (u, v) fixed, and is usually dominated by an integrable $g(x)$. It is first shown that if $y \in A$ satisfies some C_j and certain further conditions, including a one-sided restriction (below) on y'' sgn y, then $|y'(x)|\leqq H$, where H depends only on parameters and functions occurring in the hypotheses. Three theorems give sufficient conditions for the existence of at least one solution to problems P_j. In two further theorems it is assumed that

$$f(x, u, v) \operatorname{sgn} u \geqq -[p(x)|v|+q(x)|u|+r(x)]$$

where p, q, and r are non-negative and summable, and the sufficient conditions then involve the solutions of

$$w''\pm pw'+qw=0.$$

It is shown, finally, that in certain respects the hypotheses cannot be weakened. Many references occur to the Italian literature. *F. A. Ficken* (Knoxville, Tenn.).

Whyburn, William M. A nonlinear boundary value problem for second order differential systems. Pacific J. Math. 5, 147–160 (1955).

The author proves the existence of sets of characteristic numbers for nonlinear systems of the form

$$dy/dx=K(x, y, z; \lambda)z, \quad dz/dx=g(x, y, z; \lambda)y$$

with the boundary conditions $\psi(a, \lambda)=0$, $\phi(a, \lambda)=\phi(b, \lambda)$, where

$$\psi(x, \lambda)=\gamma(x, \lambda)z(x, \lambda)-\delta(x, \lambda)y(x, \lambda),$$
$$\phi(x, \lambda)=\alpha(x, \lambda)z(x, \lambda)-\beta(x, \lambda)y(x, \lambda), \quad \beta(x, \lambda)$$

does not change sign, $\delta(a, \lambda)\neq 0$ and

$$\Delta(\lambda)=\alpha(a, \lambda)\delta(a, \lambda)-\beta(a, \lambda)\gamma(a, \lambda)\neq 0$$

and gives oscillation theorems for the associated solutions. The assumptions are too long to be stated here. The method used, and the results obtained, generalize and extend those given in an earlier paper of the author [Trans. Amer. Math. Soc. 30, 848–854 (1928)]. *M. Zlámal* (Brno).

Levitan, B. M. On the asymptotic behavior of a spectral function and on expansion in eigenfunctions of a self-adjoint differential equation of second order. II. Izv. Akad. Nauk SSSR. Ser. Mat. 19, 33–58 (1955). (Russian)

The results originally proved in the author's previous paper on the same subject [same Izv. 17, 331–364 (1953); MR 15, 316] are proved again with less restrictive hypotheses. In particular, the assumption that the spectrum of the differential operator be bounded from below is dropped. The only assumption on the real function q in the differential expression $Ly=-y''+q(x)y$ is that it be integrable on each finite subinterval of $0\leqq x<\infty$. Improved asymptotic formulas for the spectral function are also obtained. *E. A. Coddington* (Copenhagen).

✶Mishoe, Luna Isaac. On the expansion of an arbitrary function in terms of the eigenfunctions of a non-self-adjoint differential system. Thesis, New York University, 1953. i+28 pp.

Consider the boundary-value problem

$$u''+q(x)u+\lambda[p(x)u-u']=0, \quad u(0)=u(1)=0.$$

Here q is continuous and p has a continuous second derivative. Let $\{u_n\}$ be eigenfunctions for this problem. It is shown that, if f is of bounded variation on $0\leqq x\leqq 1$, then the series $\sum a_n u_n$, where $a_n=\int_0^1 f(pv_n+v_n')dx$ and the v_n are appropriately normalized eigenfunctions of the adjoint problem $v''+q(x)v+\lambda[p(x)v+v']=0$, $v(0)=v(1)=0$, converges to

$$\tfrac{1}{2}[f(x+0)+f(x-0)]$$
$$-\tfrac{1}{2}\exp\left(\int_0^x pd\right)\left[f(0+)+\exp\left(-\int_0^1 pdt\right)f(1-)\right].$$

The Cauchy integral method is used together with results on the asymptotic nature of solutions of the equations involved. There are several misprints. *E. A. Coddington.*

Partial Differential Equations

✶Sobolev, S. L. Uravneniya matematičeskoĭ fiziki. [The equations of mathematical physics.] **3d ed.** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1954. 444 pp. 10 rubles.

This edition differs from the 2nd [1950; MR 13, 42] in only minor changes and corrections.

✶Germay, R. H. Sur les systèmes complètement intégrables d'équations récurrentes aux différentielles totales. IIIe Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 14–17. Fédération belge des Sociétés Scientifiques, Bruxelles.

✶Lepage, Th. Équation du second ordre et transformations symplectiques. Premier colloque sur les équations aux dérivées partielles, Louvain, 1953. pp. 79–104. Georges Thone, Liège; Masson & Cie, Paris, 1954.

This is an exposition, using exterior algebra, of results on matrix polynomials, partial differential equations and related invariants, mostly appearing in earlier papers of the author [Acad. Roy. Belg. Bull. Cl. Sci. (5) **34**, 694–708 (1949); Algèbre et théorie des nombres, Colloq. Internat. Centre Nat. Rech. Sci., no. 24, Paris, 1950, pp. 181–186; MR **11**, 308; **13**, 814]. An application to a variational problem is given. *L. C. Hutchinson* (Boston, Mass.).

✶Moisil, Gr. C. Matricele asociate sistemelor de ecuaţii cu derivate parţiale. Introducere in studiul cercetărilor lui I. N. Lopatinschi. [Associated matrices of systems of partial differential equations. Introduction to the study of the investigations of I. N. Lopatinskiĭ.] Editura Academiei Republicii Populare Române, 1950. 61 pp.

Given a system of linear and homogeneous partial differential equations with constant coefficients, the paper is concerned with setting up all possible linear homogeneous partial differential relations with constant coefficients among the unknown functions which are linear differential consequences of the given system. In terms of appropriate linear differential operators it is shown that the problem depends on a solution of the following algebraic problem: Given a linear homogeneous system of equations with coefficients which are polynomials in a certain number of variables, to determine all of its polynomial solutions. The author states, without proof, the familiar fact that if such a system has solutions, then it has a certain finite number of sets of polynomial solutions, linear combinations of which, with arbitrary polynomial coefficients, will give all polynomial solutions. For thirteen specific systems from many branches of mathematical physics the problem at hand is solved in detail by means of special devices suggested by the structure of each particular system. The reference in the title is to a paper by Lopatinskiĭ on linear differential operators [Mat. Sb. N.S. **17**(59), 267–288 (1945); MR **8**, 75]. An extensive bibliography contains also a reference to a paper by Snapper [Amer. J. Math. **69**, 622–652 (1947); MR **9**, 173].
I. J. Schoenberg (Swarthmore, Pa.).

Pucci, Carlo. Il problema di Cauchy per le equazioni lineari a derivate parziali. Consiglio Naz. Ricerche. Pubbl. Ist. Appl. Calcolo no. **371**, 3 pp. (1953).

Summary of results in Ann. Mat. Pura Appl. (4) **35**, 129–153 (1953); MR **16**, 40.

Lions, Jacques-Louis. Sur les problèmes de dérivée oblique. C. R. Acad. Sci. Paris **240**, 266–268 (1955).

In this note the author formulates a very general class of boundary-value problems for elliptic equations with variable coefficients. He starts by introducing a Hilbert norm for functions f on the boundary of a smooth domain as the square root of the infimum of the m-fold Dirichlet integral of smooth functions u having f as their kth normal derivative, $k \leqq m-1$. This norm depends only on $m-k$. Let B denote a boundary operator of order $2m-1-k$, transversal of order $m-1$; then Bu, as a boundary operator, is bounded in the above norm by the m-fold Dirichlet integral of w. Given m boundary operators $B_0, \cdots, B_{m-1}$ of the above kind, the author builds the bilinear form $(u,v)_g+\sum\langle B_k u, \gamma_k v\rangle$, where $(u,v)_g$ is a bilinear form $\sum_{|p|,|q|\leqq m}(g_{pq}D^q u, D^p v)$ in the derivatives of u, v up to order m with smooth coefficients g_{pq}, $\gamma_k v$ denotes the kth normal derivative of v and $\langle\ \rangle$ the scalar product associated with the above mentioned norm for boundary functions. This form is bounded by the m-norms of u, v. The author considers this form over some space V of functions with finite Dirichlet integrals possibly subject to some boundary conditions. The crucial property is the boundedness from below of the associated quadratic form over V. If this is the case, the boundary-value problem: $\Lambda u=f$, where $u-h$ satisfies the natural boundary conditions in the weak sense, can be solved by orthogonal projection. Hence Λ is the Euler operator associated with the quadratic form $(u,u)_g$, f any square-integrable function and h any function with finite Dirichlet integral such that Λh is square integrable. *P. D. Lax* (New York, N. Y.).

Ciliberto, Carlo. Sulle equazioni non lineari di tipo parabolico in due variabili. Ricerche Mat. **3**, 129–165 (1954).

Let T denote the rectangular region $0\leqq x\leqq X$, $0\leqq y\leqq Y$. The author treats the first boundary-value problem for equations $F(x,y,u,u_x,u_y,u_{xx})=0$ of parabolic type ($F_{u_y}\cdot F_{u_{xx}}<0$). That is, he proves, under restrictions too numerous to list here, the existence of a unique function $u=u(x,y)$ which in T is a solution of the differential equation

$$F(x,y,u,u_x,u_y,u_{xx})=0$$

and which on the boundary of T satisfies the conditions $u(x,0)=u_0(x)$, $0\leqq x\leqq X$; $u(0,y)=u_1(y)$, $u(X,y)=u_2(y)$, $0\leqq y\leqq Y$. Here u_0, u_1, u_2 are preassigned functions. For the quasi-linear equation $u_{xx}+a(x,y,u,u_x)u_y=f(x,y,u,u_x)$, $a<0$, the same problem is treated but under slightly different restrictions. The author uses a functional approach which nets existence and uniqueness of a solution but not its representation. *F. G. Dressel* (Durham, N. C.).

Gagliardo, Emilio. Formule di maggiorazione integrale per le soluzioni dell'equazione del calore non omogenea. Ricerche Mat. **3**, 202–219 (1954).

Denote by D the set of points whose x and y coordinates satisfy the inequalities

$$a\leqq y\leqq b,\quad \alpha(y)\leqq x\leqq\beta(y)\quad(\alpha(y)<\beta(y)),$$

where the functions α, β are of class C'. Let $u(x,y)$ be a function of class C^2 which in D satisfies the equation $u_{xx}-u_y=f(x,y)$ and, on the boundary of D, meets the conditions $u(x,a)=u_1(x)$ for $\alpha(a)\leqq x\leqq\beta(a)$, and

$$u(\alpha(y),y)=u_2(y),\quad u(\beta(y),y)=u_3(y)$$

for $a\leqq y\leqq b$. The function f is assumed continuous in D, the boundary values of u are continuous, and the functions u_1, u_2, u_3 have continuous first derivatives. Under these assumptions the author shows the following inequality holds

$$\iint_D[u^2+(u_x)^2+(u_y)^2+(u_{xx})^2]dxdy$$
$$\leqq h\left\{\iint_D f^2dxdy+\int_{\alpha(a)}^{\beta(a)}(u_1^2+|u_1'|^2)dx\right.$$
$$\left.+\int_a^b(u_2^2+|u_2'|^2+|u_3|^2+|u_3'|^2)dy\right\},$$

where h is a constant dependent solely on the region D.
F. G. Dressel (Durham, N. C.).

Cooke, J. C. Note on a heat conduction problem. Amer. Math. Monthly 62, 331–334 (1955).

The problem of determining the temperatures $u(r, \theta, t)$ in a wedge initially at temperature $u=1$, when $u=0$ on the boundaries $\theta=0$ and $\theta=\beta$, is used as an illustration of the directness of the method of integral transformations. The solution is obtained in a known form by applying a finite Fourier transformation and a Hankel transformation in succession. *R. V. Churchill* (Ann Arbor, Mich.).

Vodička, Václav. Circular cylinder in a periodic temperature field. Appl. Sci. Res. A. 5, 268–272 (1955).

Let $u(\rho, z, t)$ denote the steady periodic temperatures in a cylinder. The lateral surface $\rho=r$ is subjected to a linear transfer of heat into a medium whose temperature is one simple harmonic function of time t, and the bases $z=\pm s$ are subjected to linear transfer into a medium whose temperature is another simple harmonic function of t. Classical methods involving orthogonal trigonometric and Bessel functions are employed to obtain a series representation of $u(\rho, z, t)$. *R. V. Churchill* (Ann Arbor, Mich.).

★Zel'dovič, Ya. B., and Kompaneec, A. S. On the theory of propagation of heat with the heat conductivity depending upon the temperature. Sbornik posvyaščennyi semidesyatiletiyu akademika A. F. Ioffe [Collection in honor of the seventieth birthday of academician A. F. Ioffe], pp. 61–71. Izdat. Akad. Nauk SSSR, Moscow, 1950.

The heat conductivity and capacity are assumed to vary as powers of the temperature T. Consideration is restricted to the plane case (T a function of x and t) and the spherical case (T a function of r and t). The respective differential equations are found to be

$$\partial T/\partial t=a(\partial/\partial x)(T^n\partial T/\partial x)$$

and

$$\partial T/\partial t=(ar^{-2})(\partial/\partial r)(r^2T^n\partial T/\partial r),$$

with the supplementary conditions

$$\int_{-\infty}^{\infty} T\,dx=\text{const.},\quad \int_{-\infty}^{\infty} Tr^2\,dr=\text{const.}$$

Separation of the variables leads to equations admitting a one-parameter transformation group. Explicit solutions are found for the plane case. *F. V. Atkinson.*

Sobolev, S. L. On a new problem of mathematical physics. Izv. Akad. Nauk SSSR. Ser. Mat. 18, 3–50 (1954). (Russian)

The author solves the initial-value problem for the following system of partial differential equations:

$$(*)\qquad \frac{\partial \mathbf{V}}{\partial t}-\mathbf{V}\times\mathbf{k}+\operatorname{grad} p=\mathbf{F},\quad \operatorname{div}\mathbf{V}=g.$$

Here $\mathbf{F}$ is a given vector, g is a given function and $\mathbf{k}$ is a unit vector in the z-direction. The vector $\mathbf{V}$ and the function p are to be determined subject to the initial condition at time $t=0$ that $\mathbf{V}$ equal a given vector $\mathbf{V}_0$ for all points in a region R which may be the whole space or may be bounded by a closed surface S. In the latter case a boundary condition, such as $p=0$ on S or the normal component of $V=0$ on S, must be satisfied. The author uses the method of orthogonal projection in Hilbert space to show that (*) has a solution. If R is the whole space, a fundamental solution of the system is constructed and then by the use of generalized Green's formula an explicit solution of (*) is obtained. Finally, the author obtains an explicit solution of $\partial^2\Delta u/\partial t^2+\partial^2u/\partial z^2=\phi$ with the initial conditions: $u=u_0$ and $\partial u/\partial t=u_1$ at time $t=0$. *B. Friedman* (Berkeley, Calif.).

Sobolev, S. L. On a new problem of mathematical physics. Acad. Repub. Pop. Romîne. An. Romîno-Soviet. Mat.-Fiz. (3) 9, no. 1(12), 5–55 (1955). (Romanian)

Translation of the paper reviewed above.

Capriz, Gianfranco. Sulla applicazione del metodo della trasformata parziale di Laplace ad intervallo di integrazione finito ad un problema di elasticità piana. Ann. Scuola Norm. Super. Pisa (3) 7, 17–41 (1953).

In the first part of this paper, the author considers the problem of finding two functions $u(x, y)$ and $v(x, y)$ which satisfy the following pair of partial differential equations:

$$\Delta_2 u+\sigma\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\frac{1}{\mu}f(x, y),$$

$$\Delta_2 v+\sigma\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\frac{1}{\mu}g(x, y),$$

in a region T defined by $0\leqq x\leqq 1$, $\alpha(x)\leqq y\leqq\beta(x)$, and which assume prescribed values along the boundaries of T. Using transformations of the form $u^*(x, q)=\int_{\alpha(x)}^{\beta(x)} e^{qy}u(x, y)\,dy$, the problem is formally reduced to a pair of second-order ordinary differential equations in u^* and v^*. With prescribed values at $x=0$ and $x=1$. These equations are solved, but the solution involves, besides the given boundary values of u and v, functions which depend on the unknown values of u_x, u_y, v_x, v_y on the boundary curves. However, since the transforms u^* and v^* are analytic functions of q, the residues at the points which appear to be poles of these functions must vanish, and this condition leads to the formulation of an infinite number of integral equations which must be solved to give the unknown functions. In the third part of the paper, the author shows the conditions under which the solution of this set of integral equations is equivalent to the given problem, using results of G. Fichera. The second part of the paper is devoted to a discussion of the same pair of equations in the strip S defined by $0\leqq x\leqq 1$, $-\infty<y<+\infty$, where both functions are to vanish on the boundary of S. Using Fourier transforms, the author finds conditions under which the resulting pair of ordinary differential equations will lead to a unique solution of the problem, if existence of a solution is assumed, and also conditions under which a generalized solution exists. *D. L. Bernstein.*

Blondel, Jean-Marie. Sur une classe d'équations aux dérivées partielles linéaires. C. R. Acad. Sci. Paris 240, 1181–1183 (1955).

Let $P(x, y; u, v)$ be a polynomial of degree $\leqq n-1$ in u and v separately with coefficients of class C^∞ in x and y. Let $\Phi(x, y)$, $f_i(x)$, and $g_i(y)$ $(i=0, 1, \cdots, n-1)$ be summable. $z(x, y)$ is sought such that

$$\frac{\partial^{2n}z}{\partial x^n\partial y^n}+P\left(x, y; \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)z=\Phi(x, y),$$

$$\frac{\partial^i z(x, 0)}{\partial y^i}=f_i(x),\quad \frac{\partial^i z(0, y)}{\partial x^i}=g_i(y).$$

It is stated that by conversion to a Volterra integral equation an existence and uniqueness theorem can be proved. A fundamental solution is obtained, and explicit results are presented in certain special cases. *F. A. Ficken.*

Functional Analysis

Collins, Heron Sherwood. Completeness and compactness in linear topological spaces. Trans. Amer. Math. Soc. **79**, 256–280 (1955).

Ce travail se divise en trois parties. Dans la première, l'auteur apporte quelques compléments aux résultats connus de Šmulian, Eberlein et Grothendieck sur la compacité faible dans un espace localement convexe X; un ensemble $M\subset X$ est dit faiblement compact en moyenne (notion due à Šmulian) si pour toute suite (x_n) dans M, il existe $x \in X$ tel que $\lim\inf_{n\to\infty} f(x_n) \leqq f(x) \leqq \lim\sup_{n\to\infty} f(x_n)$ pour toute $f \in X'$ (dual de X). L'auteur montre que si X est complet pour la topologie de Mackey $\tau(X, X')$, un ensemble faiblement compact en moyenne est relativement faiblement compact dans X. Dans la seconde partie, l'auteur considère les topologies sur X' qui sur toute partie équicontinue, induisent la même topologie que la topologie faible $\sigma(X', X)$. Il montre qu'en général la plus fine de ces topologies n'est pas localement convexe, et que la plus fine des topologies localement convexes ayant cette propriété n'est pas une $\mathfrak{S}$-topologie (i.e. une topologie de convergence uniforme sur une famille de parties bornées de X). Enfin, la troisième partie est une étude de la notion d'espace X pleinement complet, un espace ayant par définition cette propriété si, dans X', toute variété linéaire, dont l'intersection avec tout polaire U^0 d'un voisinage de 0 dans X est fermée pour la topologie faible $\sigma(X', X)$, est elle-même fermée pour cette topologie. Il est connu que si X est pleinement complet, il est complet, et la réciproque est vraie si X est métrisable, mais non en général. L'auteur montre que tout sous-espace fermé d'un espace pleinement complet est pleinement complet, ainsi que tout espace quotient par un sous-espace fermé; mais ni les produits ni les sommes directes topologiques d'espaces pleinement complets ne sont nécessairement pleinement complets. *J. Dieudonné.*

Klee, V. L., Jr. Boundedness and continuity of linear functionals. Duke Math. J. **22**, 263–269 (1955).

Soient E un espace vectoriel topologique métrisable sur le corps R des nombres réels, C une partie convexe de E (apparemment, il faut $0 \in C$). Les propriétés suivantes sont équivalentes: (i) il existe un sous-espace vectoriel fermé L de codimension finie dans E tel que: (a) $C\subset L$; (b) $C-C$ possède relativement à L un intérieur non vide; (ii) toute forme linéaire sur E bornée sur C est continue; (iii) (si E est complet et C fermé) toute forme linéaire f sur E telle que $f(C)\neq R$ est continue. Extension au cas complexe. Exemple d'application: sur l'espace L^p $(0<p<1)$, une forme linéaire $\geqq 0$ est identiquement nulle. *J. Dixmier* (Paris).

Klee, V. L., Jr. Some topological properties of convex sets. Trans. Amer. Math. Soc. **78**, 30–45 (1955).

The author continues his studies on convex bodies. Among the results obtained are the following. A compact convex subset of a normed linear space is homeomorphic with a parallelotope. Thus all infinite-dimensional separable Banach spaces are of the same dimension type (in the terminology of Fréchet). If E is a locally convex metrizable topological linear space and K is a non-compact subset of E, then K lacks the fixed-point property. (This result is related to one of Tychonov which states that if K is compact and convex, then K has the fixed-point property.) Let E be an infinite-dimensional Banach space, X a compact subset of E, and f a homeomorphism of X into E. If E is a Hilbert space or if X is finite-dimensional, there is an isotopy μ of E onto E such that μ_0 is the identity and $\mu_1|X=f$. The next result is based on methods introduced by O. H. Keller. Suppose K is an infinite-dimensional compact convex subset of a normed linear space and $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_n\}$ are sets of distinct points of K. Then there is a homeomorphism h of period 2 of K onto K such that $hx_i=y_i$. Finally there is a section on topological representatives of convex sets. If C is a locally compact closed convex subset of a normed linear space, then there are cardinal numbers m and n, $0\leqq m\leqq\aleph_0$, $0\leqq n\leqq\aleph_0$, such that C is homeomorphic with either $[0, 1]^m\times]0, 1[^n$ or $[0, 1]^m\times[0, 1[$. The various possibilities are topologically distinct. *E. R. Lorch.*

Shirota, Taira. On completely continuous operators on locally convex vector spaces. Proc. Japan Acad. **30**, 837–840 (1954).

This paper proves that if t is a completely continuous transformation on a linear convex topological space X to X in the sense that there exists a vicinity U of 0 such that $t(U)$ is relatively (bi)-compact then the transposed transformation t^* on the space X^* of linear continuous functionals on X is completely continuous in the compact open topology on X^*. Consequently, in view of the paper of J. Leray [Acta Sci. Math. Szeged **12**, Pars B, 177–186 (1950); MR **12**, 32], the Fredholm theorems apply to $1-\lambda t$ and $1^*-\lambda t^*$ and their interrelations. M. Altman [Studia Math. **13**, 194–207 (1953); MR **15**, 436] has proved a similar result using sequential compactness and a topology based on bounded sets in X. [See also J. H. Williamson, J. London Math. Soc. **29**, 149–156 (1954); MR **15**, 801.] If t is completely continuous then $t^*\cdot t^*$ is completely continuous in the strong topology on X^*, leading to Fredholm alternatives for a t for which there exists an integer n such that t^n is completely continuous. *T. H. Hildebrandt* (Ann Arbor, Mich.).

Nikaidô, Hukukane. A proof of the invariant mean-value theorem on almost periodic functions. Proc. Amer. Math. Soc. **6**, 361–363 (1955).

The now classical result on the existence of an invariant mean for almost periodic functions on a group G is derived from the following fixed-point theorem: Let K be a compact convex subset of a locally convex, topological linear space E. Let φ be a continuous mapping of K into itself for which $\varphi(\alpha_1x_1+\alpha_2x_2)=\alpha_1\varphi(x_1)+\alpha_2\varphi(x_2)$, $0\leqq\alpha_i$, $\alpha_1+\alpha_2=1$, $x_i \in K$. Then there is an $x \in K$ such that $\varphi(x)=x$. From this result the author deduces the invariant mean theorem in the form: Let E^G be the set of continuous mappings of G into E. For each $\varphi \in E^G$, let $L_\varphi=\{\varphi(ax)|a \in G\}$, and let K_φ be the convex closure of L_φ (in the weak* topology of E^G). Then for each $\varphi \in E^G$, K_φ contains a constant function (the invariant mean of φ). In particular the existence of Haar measure follows from this. *B. Gelbaum* (Minneapolis, Minn.).

Iséki, Kiyoshi. Vector-space valued functions on semigroups. I. Proc. Japan Acad. **31**, 16–19 (1955).

The Bochner-von Neumann theory of almost periodic functions from groups into vector spaces is partially duplicated here in the context where "semigroups" replace "groups". The paper treats "ergodic" functions. ($f(x)$ is ergodic (vector-valued) if the means of its translates converge to a constant (vector), i.e., for each neighborhood U of the (vector) origin, there is a set $a_1, \cdots, a_n$ in the semigroup G and a vector f such that $f-n^{-1}\sum_1^n f(a_id) \in U$ for all d in G. f is called a "left (ergodic) U-mean". Similarly, for a "right V-mean", g.) Theorem: $f-g \in U+V$. Corollary: If the vector space is more than locally convex, namely a

Banach space or a complete locally convex space, then $f(x)$ (ergodic) has a unique mean.
B. Gelbaum (Minneapolis, Minn.).

Miyadera, Isao. A note on strongly (C, α)-ergodic semi-group of operators. Proc. Japan Acad. **30**, 797–800 (1954).

This paper is concerned with the strong (C, α) convergence to the identity at zero of a semi-group of linear bounded operators. The author proves the following theorem: Let α be a positive integer. A necessary and sufficient condition that a semi-group of operators strongly Abel-ergodic to the identity at zero be strongly (C, α)-ergodic to the identity at zero is that there exist a positive number M such that

$$\sup_{\lambda>0,\, k\geqq\alpha}\Bigg\|\frac{\alpha}{k(k-1)\cdots(k-\alpha+1)} \times \sum_{i=1}^{k-\alpha+1} \frac{(k-i)!}{(k-\alpha+1-i)!}[\lambda R(\lambda; A)]^i\Bigg\| \leqq M.$$

This generalizes the $(C, 1)$ case previously established by the author [Tôhoku Math. J. (2) **6**, 38–52 (1954); MR **16**, 374] and the reviewer [Ann. of Math. (2) **59**, 325–356 (1954); MR **15**, 718]. It is worth noting that the principal identity (7) of this paper can be derived more directly by writing $\xi^{k-\alpha}$ in line 20, p. 798, as $[(\xi-\tau)+\tau]^{k-\alpha}$ and expanding the binomial. *R. S. Phillips* (Los Angeles, Calif.).

Miyadera, Isao. On the generation of strongly ergodic semi-groups of operators. II. Tôhoku Math. J. (2) **6**, 231–242 (1954).

A semi-group of linear bounded operators $[T(\xi); \xi>0]$ on a complex Banach space X to itself, strongly measurable for $\xi>0$ with $\int_0^1 \|T(\xi)x\|d\xi<\infty$ for each $x \in X$, is said to be of class $(0, A)$ if $\lim_{\lambda\to\infty} \lambda\int_0^\infty \exp(-\lambda\xi)T(\xi)x\, d\xi=x$, $x \in X$, and of class $(0, C_\alpha)$ if $\lim_{\tau\to0} \alpha\tau^{-\alpha}\int_0^\tau (\tau-\xi)^{\alpha-1}T(\xi)x\, d\xi=x$, $x \in X$. It was shown by the reviewer [Ann. of Math. (2) **59**, 325–356 (1954); MR **15**, 718] that the infinitesimal generator of such semi-groups has a smallest closed extension A, called the complete infinitesimal generator (c.i.g.), whose resolvent $R(\lambda; A)$ is the Laplace transform of $T(\xi)$ for $\Re(\lambda)>\lim_{\xi\to\infty} \xi^{-1}\log\|T(\xi)\|$. Without loss of generality one may suppose that $\int_0^\infty \|T(\xi)x\|d\xi<\infty$, $x \in X$. The following generation theorem is proved: A necessary and sufficient condition that a closed linear operator A is the c.i.g. of a semi-group $[T(\xi)]$ of class $(0, A)$ with $\int_0^\infty \|T(\xi)x\|d\xi<\infty$, $x \in X$, is that (i) the spectrum of A is contained in $[\lambda; \Re(\lambda)\leqq0]$, (ii) the domain of A is dense in X, (iii) $\|R(\lambda; A)\|=O(1/\lambda)$ as $\lambda\to\infty$, and (iv) there exists a non-negative function $f(\xi, x)$ defined on $(0, \infty)\times X$ such that $f(\xi, x)$ is continuous and integrable on $(0, \infty)$ for each $x \in X$ and $\|R^{(k)}(\lambda; A)x\|\leqq(-1)^kF^{(k)}(\lambda, x)$ for each $x \in X$, all real $\lambda>0$, and all integers $k\geqq0$, where

$$F(\lambda, x)=\int_0^\infty \exp(-\lambda\xi)f(\xi, x)d\xi.$$

In this case $\|T(\xi)x\|\leqq f(\xi, x)$ and

$$T(\xi)x=\lim_{\lambda\to\infty}\exp[\xi(-\lambda+\lambda^2R(\lambda; A)]x$$

for each $x \in X$ and all $\xi>0$. Combining the above theorem with a result obtained in the paper reviewed above, the author is able to state a necessary and sufficient condition that a closed linear operator be the c.i.g. of a semi-group of class $(0, C_\alpha)$ for α a positive integer. *R. S. Phillips*.

Baher, F. S. On a basis in the space of continuous functions defined on a compactum. Dokl. Akad. Nauk SSSR (N.S.) **101**, 589–592 (1955). (Russian)

The problem of establishing a basis for the Banach space $C(Q)=\{f(x)\,|\,f(x)$ continuous on $Q\}$, Q a compactum, is solved. The techniques are reminiscent of Schauder's original construction for the case $Q=[0, 1]$. The decisive steps are: (a) the mapping of Q onto a subset of the Hilbert cube; (b) canonical decompositions of "elementary figures" (those bounded by a finite set of hyperplanes in Hilbert space); (c) the successive "squeezing" of Q (considered in the Hilbert cube) into a sequence of strips (point sets bounded by pairs of hyperplanes); (d) constructing analogues of Schauder's piecewise linear functions (which are a basis for $C[0, 1]$) by the use of the elementary figures and polyhedral sets which arise in (c) above. Applications of the result are given for abelian normed rings which, by virtue of theorems of Gelfand and his school, are essentially the totality of continuous functions over the spaces of their maximal ideals. Only outlines of proof are offered.
B. R. Gelbaum (Minneapolis, Minn.).

Franckx, E. Convergence faible des variables vectorielles bornées. Assoc. Actuair. Belges. Bull. no. **57**, 37–55 (1954).

Expository paper. *K. L. Chung* (Syracuse, N. Y.).

Maharam, Dorothy. On kernel representation of linear operators. Trans. Amer. Math. Soc. **79**, 229–255 (1955).

F'-integrals (linear, countably additive, order-preserving mappings of a real function space F, modulo null functions, into another, F', both satisfying the countable chain condition) were introduced previously by the author [same Trans. **75**, 154–184 (1953); MR **14**, 1071]. In this paper generalizations of the Radon-Nikodym theorem for such integrals are obtained. An F'-integral φ is said to be full-valued if $\varphi(F^+)$ includes with any function g all non negative functions $h\leqq g$. Any F'-integral φ has a minimal full-valued extension φ^*. It is shown that if φ is full-valued then another F'-integral ψ admits a kernel representation $\psi(f)=\varphi(kf)$, where k is a fixed non-negative function, if and only if φ dominates ψ in the sense that the support of $\varphi(f)$ includes that of $\psi(f)$ for all $f \in F^+$. An example shows that when φ is not full-valued this kind of dominance is not sufficient, and does not even insure that ψ admits a kernel representation in terms of φ^*. However, a necessary and sufficient condition that $\psi(f)=\varphi^*(kf)$ is that φ dominate ψ in a stronger sense: namely, $\psi=\sum\psi_n$ where $\psi_n(f)\leqq\varphi(f)$ on F^+. F'-operators of bounded variation (differences of two F'-integrals) are also studied, and generalizations of the Hahn decomposition and kernel representation, with k finite but no longer positive, are obtained. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Krabbe, G. L. The Titchmarsh semi-group. Proc. Amer. Math. Soc. **6**, 219–225 (1955).

With $p>1$, let S_p denote the space of all complex sequences $a=\{a_\nu\}_{\nu=-\infty}^{\infty}$ with $\|a\|_p=(\sum_{\nu=-\infty}^{\infty}|a_\nu|^p)^{1/p}<\infty$. With complex α, the transformation T_α is defined by

$$[T_\alpha a]_n=\sum_{\nu=-\infty}^{\infty}(-1)^{n+\nu}(\sin\alpha\pi)\{(n+\alpha-\nu)\pi\}^{-1}a_\nu \qquad (n=0, \pm1, \cdots).$$

The transformation G is defined by

$$[Ga]_n=\sum_{\nu=-\infty}^{\infty}(-1)^{n+\nu}(n-\nu)^{-1}a_\nu,$$

$\xi \neq n$. It has been proved by M. Riesz [Math. Z. 27, 218–244 (1927)] that T_α and G are bounded linear operators in S_p. It is here proved that: (i) T_α is an entire function of α; (ii) $T_0 a = a$ and $T_\alpha T_\lambda a = T_{\alpha+\lambda} a$ for all complex α, λ and all $a \in S_p$. The relations (ii), which were established by Titchmarsh [ibid. 25, 321–347 (1926)] in the case $\alpha=\frac{1}{2}$, $\lambda=-\frac{1}{2}$ and by M. Riesz [loc. cit.] in the case $p=2$, show that $\{T_\alpha|\alpha\}$ is a group, here called the Titchmarsh semi-group. It is further proved that: (iii) G is the infinitesimal generator of the Titchmarsh semi-group. These results are obtained by proving that there is a non-decreasing function f such that, for all α, $\|T_\alpha\|_p \leq f(|\operatorname{Im} \alpha|)$. *F. F. Bonsall.*

Iohvidov, I. S. On the theory of indefinite Toeplitz forms. Dokl. Akad. Nauk SSSR (N.S.) 101, 213–216 (1955). (Russian)

The author considers a unitary operator U in an infinite-dimensional space H_k with an indefinite metric defined by a Hermitian form

$$(x, x) = |\xi_1|^2 + \cdots + |\xi_k|^2 - |\xi_{k+1}|^2 - |\xi_{k+2}|^2 - \cdots$$

with exactly k positive terms. It is known [I. S. Iohvidov, same Dokl. (N.S.) 71, 225–228 (1950); MR 12, 33] that there is a k-dimensional invariant subspace of U corresponding to k eigenvalues λ_j $(1 \leq j \leq k)$ such that $|\lambda_j| \geq 1$. Forming the operator $A = -S^*S = -SS^*$, where

$$S = \prod_{j=1}^{k} (U - \lambda_j I),$$

the author shows that $(Ax, x) \geq 0$ for all $x \in H_k$. He then constructs a finite-difference operator L, depending only on the λ_j, such that, if $c_p = (U^p x, x)$ $(p = 0, \pm 1, \cdots)$, then $L(c_p) = \alpha(AU^p x, x)$, where α is a certain positive real constant. Using the definiteness of the form (Ax, x) to construct a Hilbert-space metric in a quotient space of H_k, he obtains an integral representation

$$(1) \qquad L(c_p) = \int_{-\pi}^{\pi} e^{ipt} d\sigma(t) \quad (p = 0, \pm 1, \cdots),$$

where $\sigma(t)$ is a non-decreasing function, continuous on the left. He then solves the difference equation (1) to obtain an explicit but rather complicated integral representation for the sequence (c_p) itself.

The following result is obtained as a by-product. Let (c_p) $(p = 0, \pm 1, \cdots)$ be a sequence such that c_0 is real, $c_{-p} = \bar{c}_p$ and, for all sufficiently large n, the Toeplitz form

$$\sum_{p,q=0}^{n} c_{p-q} \xi_p \bar{\xi}_q$$

contains exactly k positive squares when reduced to its canonical form. Then there exists a linear finite-difference operator L of exact order $2k$, containing only differences of even order, such that the sequence $L(c_p)$ is non-negative definite, i.e., the Hermitian form

$$\sum_{p,q=0}^{n} L(c_{p-q}) \xi_p \bar{\xi}_q$$

is non-negative definite.

Only outline proofs are given. *F. Smithies.*

Kasahara, Shouro. A characterization of Hilbert space. Proc. Japan Acad. 30, 846–848 (1954).

The author gives as a sufficient condition for the norm in normed linear space E to be derived from an inner product the following weakening of the von Neumann-Jordan condition: There is a positive number $\alpha \leq \frac{1}{2}$ such that for each pair x, y of elements of E there is a number $\lambda = \lambda(x, y)$ such that $\alpha \leq \lambda \leq 1 - \alpha$ and

$$(*) \quad \lambda\|x\|^2 + (1-\lambda)\|y\|^2 \geq \lambda(1-\lambda)\|x-y\|^2 + \|\lambda x + (1-\lambda)y\|^2.$$

(If $\|\ \|$ in E is defined from an inner product, it is clear that $=$ holds for all x, y, and λ.) The author gives a neat proof that this condition implies the Kakutani criterion for an inner-product space: Every closed linear subspace of E has a projection on it of norm one [Jap. J. Math. 16, 93–97 (1939); MR 1, 146].

As Schoenberg [Proc. Amer. Math. Soc. 3, 961–964 (1952); MR 14, 564] showed of another generalization of the von Neumann-Jordan criterion, the method of the reviewer [Trans. Amer. Math. Soc. 62, 320–337 (1947); MR 9, 192] of inscribing or circumscribing an ellipse in the unit sphere of a two-dimensional section of E can be used to prove this sufficient even if α is dropped and it is assumed only that for each x, y with $\|x\| = \|y\| = 1$ there is a λ, $0 < \lambda < 1$, for which (*) holds. *M. M. Day.*

Naĭmark, M. A. On some criteria of completeness of the system of eigen and adjoint vectors of a linear operator in Hilbert space. Dokl. Akad. Nauk SSSR (N.S.) 98, 727–730 (1954). (Russian)

Two general results are proved concerning the completeness of the system of eigen and adjoint vectors of non-self-adjoint operators having discrete spectra.

Theorem 1: Let A be a closed linear operator, defined on a dense linear space in the Hilbert space H, whose resolvent R_λ is completely continuous. Let there exist a dense subset S of H and a sequence of circumferences Γ_n with centers at the origin having the properties: (1) on Γ_n there is no eigenvalue of A, (2) the radii of the Γ_n tend to infinity, (3) for all $f \in S$ the resolvent R_λ satisfies

$$\max_{\lambda \in \Gamma_n} \|R_\lambda f\| \to 0 \quad \text{as} \quad n \to \infty.$$

Then the system of eigen and adjoint vectors of A are complete in H.

Theorem 2: Let A, Γ_n, S be the same as in Theorem 1, and let B be a linear operator in H such that: (1) $S \subset D_A \cap D_B$, where D_A, D_B are the domains of A and B; (2) for all sufficiently large n and $\lambda \in \Gamma_n$, (a) $(A + B - \lambda 1)S$ is dense in H, (b) $\|R_\lambda B f\| \leq q\|f\|$ for all $f \in S$, where $q < 1$. Then $A + B$ has a closure C, the resolvent of C is completely continuous, and the system of eigen and adjoint vectors of C is complete in H.

These results are applied to the case of singular differential operators to obtain the following result: Let q be an essentially bounded measurable complex-valued function on $0 \leq x < \infty$, θ a real number. For $k > 2$ the spectrum of the operator L, generated by the differential expression $l(y) = -y'' + (x^k + q(x))y$ and the boundary condition $y'(0) - \theta y(0) = 0$, is discrete. The resolvent of L is completely continuous, and the set of eigen and adjoint vectors of L are complete in $L_2(0, \infty)$. [For definitions and previous results of this nature see M. V. Keldyš, same Dokl. (N.S.) 77, 11–14 (1951); MR 12, 835.] *E. A. Coddington.*

Fourès, Y., and Segal, I. E. Causality and analyticity. Trans. Amer. Math. Soc. 78, 385–405 (1955).

The authors extend substantially the theory of Paley and Wiener [Fourier transforms in the complex domain, Amer. Math. Soc. Colloq. Publ., v. 19, New York, 1934] and Bochner [Amer. J. Math. 59, 732–738 (1937)] of functions analytic in a tube to operator-valued functions. Let H be

the Hilbert space of all square-integrable functions from real n-space G to a complex separable Hilbert space K with the scalar product $(f, g) = \int_G (f(a), g(a))_K da$, where da denotes the element of Lebesgue measure on G. A linear closed and densely defined operator S on H is called homogeneous if it commutes with translations in G of the elements of H. It is shown that such an operator is of the form $L^{-1}TL$ where L denotes Fourier transformation and T multiplication by a function T defined on the dual of G whose values are closed linear operators on K. This function is called the gain function of S. When C is a proper cone in G, S is said to be causal with respect to C if Sf vanishes in $a-C$ whenever f does (a arbitrary in G). When S is bounded, a necessary and sufficient condition for causality is that the gain function of S have a bounded analytic extension to the complex tube Γ whose base is the dual of C. With some modifications, this result is true also when S is not bounded. In the special case when T is the inverse of a polynomial p (and S a Green's operator for the corresponding differential operator), S turns out to be causal with respect to C if and only if p has no zeros in the (open) tube Γ. Finally it is shown that when S is bounded, then it has a domain of dependence in the usual sense. *L. Gårding* (Lund).

Kadison, Richard V. On the additivity of the trace in finite factors. Proc. Nat. Acad. Sci. U. S. A. **41**, 385–387 (1955).

Murray and von Neumann [Ann. of Math. (2) **37**, 116–229 (1936); Trans. Amer. Math. Soc. **41**, 208–248 (1937)] established the existence and uniqueness of a trace function on certain rings of operators on Hilbert space. One of the most difficult steps in their proofs was to show the additivity of the function in question. The present author outlines an argument for factors of type II_1, which is in part a modification of the argument of Murray and von Neumann, but in which the additivity is proved in a more elementary manner. *F. I. Mautner* (Princeton, N. J.).

Stinespring, W. Forrest. Positive functions on C^*-algebras. Proc. Amer. Math. Soc. **6**, 211–216 (1955).

Soient A et B des C^*-algèbres, μ une application linéaire de A dans B. On dit que μ est positive si $A \geqq 0$ entraîne $\mu(A) \geqq 0$. Soit $A^{(n)}$ l'algèbre des matrices de degré n à éléments dans A. En appliquant μ à chaque élément d'une matrice de A, on définit une application $\mu^{(n)}$ de $A^{(n)}$ dans $B^{(n)}$. On dit que μ est complètement positive si $\mu^{(n)}$ est positive pour tout n. Soient alors A une C^*-algèbre à élément unité, et μ une application linéaire de A dans l'algèbre des opérateurs d'un espace hilbertien H, avec $\mu(1)=1$. La positivité complète de μ est la condition nécessaire et suffisante pour que H puisse se plonger dans un espace hilbertien K, avec $\mu(S)P_H = P_H\rho(S)P_H$, ρ étant un $*$-homomorphisme de A dans l'algèbre des opérateurs de K. Quand A est commutative, la positivité complète équivaut à la positivité; le théorème est donc une généralisation d'un théorème de Neumark [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **41**, 359–361 (1943); MR **6**, 71]. *J. Dixmier* (Paris).

Yen, Ti. Trace on finite AW^*-algebras. Duke Math. J. **22**, 207–222 (1955).

Soit A une AW^*-algèbre. §2: la décomposition polaire usuelle des opérateurs est établie pour les éléments de A. §3: A est une W^*-algèbre s'il existe une famille (φ_α) de formes linéaires positives sur A, unitairement invariante, totale (i.e., pour tout $a \neq 0$ de A, il existe α avec $\varphi_\alpha(a) \neq 0$), et telle que les boules de A soient complètes pour la topologie définies par les semi-normes $\varphi_\alpha(x^*x)^{1/2}$. §4: Soit (P) la condition: il existe une famille totale de formes linéaires positives complètement additives sur A. L'A. prouve que, si (P) est remplie, A, supposée commutative, est une W^*-algèbre. Ce résultat était connu, et vient d'être établi par Kadison sans l'hypothèse de commutativité (non publié). Ce résultat de Kadison diminue la portée de certains résultats des §§5–6 (établis pour des AW^*-algèbres satisfaisant à (P)). Notons toutefois: soit A une AW^*-algèbre de type II_1, de centre Z; s'il existe une application linéaire positive idempotente et complètement additive de A sur Z, alors A possède une trace à valeurs dans Z. §7: soit Z une AW^*-algèbre commutative; il existe une AW^*-algèbre de type II, de centre Z, qui admet une trace. *J. Dixmier* (Paris).

Blum, E. K. A theory of analytic functions in Banach algebras. Trans. Amer. Math. Soc. **78**, 343–370 (1955).

The study is centered about the behavior of an analytic function in the neighborhood of a singularity. After giving a resumé of the known theory of analytic functions over a Banach algebra, the author establishes the Laurent expansion for a function $f(z)$ analytic in an annular region D which surrounds a point z_0. The nature of this region has to be specified in detail. Here it contains a ring in the $(z_0+\lambda g)$-plane where g is a fixed element for which the inverse exists and λ is a variable complex scalar. The extent of the region of validity of the expansion is considered. By means of the methods of maximal-ideal theory a region of validity is obtained which, however, is not maximal, as the author shows by examples. The definition of a singularity at a point with reference to a $(z_0+\lambda g)$-plane is now apparent. This introduces poles and essential singularities. It is shown that if B is an algebra without radical and in which the regular elements are dense, then every sphere about a plane singularity of $f(z)$ contains a point (infinitely many) not in the domain of analyticity of $f(z)$. For the same type of algebra, it is shown that at a pole b, $\lim_{z\to b} \|f(z)\| = \infty$ regardless of the manner of approach to b. Next the author considers polynomials $p(x) = a_0 + \cdots + a_n x^n$. If $D(p)$ denotes the set of elements x for which $[p(x)]^{-1}$ exists and $S(p)$ is the complement of $D(p)$, some results are obtained concerning these sets. Thus if some a_i, $i<n$, is not in the radical of the ring, $S(p)$ contains a regular element. There is a discussion by means of examples of the difficulties which are encountered in an attempt to generalize the fundamental theorem of algebra. The last section is devoted to rational functions of the second degree and in particular to the function $f(x) = x^{-1}(x-c)^{-1}$. The principal result is: If the radical of the algebra B is $\{0\}$ and if d is a residue of $f(x)$, then d is of the form $2\pi i \sum_{j=1}^{p} n_j c_j e_j$, where $e_j^2 = e_j$, $e_j e_k = 0$ for $j \neq k$, and $c_j c = e_j$ with $c(M) \neq 0$ for M in $\bigcup_{j=1}^{p} \mathfrak{M}_{1j}$. Here M represents a maximal ideal and $\mathfrak{M}_{1j} = \{M \mid e_j(M) = 1\}$. Conversely, if $c(M) \neq 0$ for all M in $\bigcup_{j=1}^{p} \mathfrak{M}_{1j}$, then there is a curve K in the intersection of the set G of regular elements and its translate $c+G$ such that $\int_K f(x)\,dx = d$, where $cd = 2\pi i \sum_{j=1}^{p} n_j e_j$. *E. R. Lorch* (New York, N. Y.).

Theory of Probability

Muller, Maurice. La notion de probabilité et ses applications. Mitt. Verein. Schweiz. Versich.-Math. **55**, 35–56 (1955).

Some miscellaneous comments concerning the nature of probability, not sufficiently unified to be briefly summarised.

The author states that past discussions have always been a function of the philosophical climate of the times. For example, the Kantian idea of synthetic a priori mental activity is unfashionable in relation to probability. Some physical and actuarial examples are considered and it is emphasised that probability is not simply a frequency.

I. J. Good (Cheltenham).

Baxter, Glen. On a characterization of the normal law. Proc. Nat. Acad. Sci. U. S. A. **41**, 383–385 (1955).

A new proof, using infinite divisibility, is given of Linnik's recent theorem on the characterization of the normal distribution [Dokl. Akad. Nauk SSSR (N.S.) **83**, 353–355 (1952); **89**, 9–11 (1953); MR **14**, 60; **15**, 42]. An example is given to show that an analogous statement for a stable distribution is false. *K. L. Chung* (Syracuse, N. Y.).

Lukacs, Eugene. A characterization of the gamma distribution. Ann. Math. Statist. **26**, 319–324 (1955).

The gamma distribution is defined as having density 0 for $x \leqq 0$ and density $\alpha^\lambda(\Gamma(\lambda))^{-1}x^{\lambda-1}e^{-\alpha x}$ otherwise, where $\lambda>0$ and α is the scale parameter. The author characterizes the gamma distribution by proving the following theorem. Let X and Y be two nondegenerate and positive random variables, and suppose that they are independently distributed. The random variables $U=X+Y$ and $V=X/Y$ are independently distributed if and only if both X and Y have gamma distributions with the same scale parameter. [The author wishes to state that the references to E. J. G. Pitman's work are not quite correct in so far as Pitman did not assume that the random variables are necessarily identically distributed.] *J. Wolfowitz* (Ithaca, N. Y.).

Blanc-Lapierre, André, et Fortet, Robert. Sur les répartitions de Poisson. C. R. Acad. Sci. Paris **240**, 1045–1046 (1955).

"The most general definition" of a Poisson distribution is given in an abstract space. Several properties are stated which reduce to well-known ones in the usual case.

K. L. Chung (Syracuse, N. Y.).

Fisz, M., and Urbanik, K. The analytical characterization of the composed non-homogeneous Poisson process. Bull. Acad. Polon. Sci. Cl. III. **3**, 149–150 (1955).

The authors state without proof a condition, too complicated to be quoted here, which assures that a process with independent increments is a non-homogeneous compound Poisson process. The characteristic function of this process is expressed in terms of a function used in formulating this condition. *E. Lukacs* (Washington, D. C.).

Fisz, M. Accuracy of an asymptotical formula. Zastos. Mat. **2**, 62–66 (1954). (Polish. Russian and English summaries)

In connection with the problem of approximating the Poisson distribution by the normal distribution the author considers the expressions

$$u_r=\exp\{-\lambda\}\lambda^r/r!, \quad v_r=\exp\{-\tfrac{1}{2}t^2\}(2\pi\lambda)^{-1/2},$$

and

$$w_r=\exp\{-\tfrac{1}{2}t^2\}(2\pi\lambda)^{-1/2}(1-\tfrac{1}{2}t\lambda^{-1/2}+t^3\lambda^{-1/2}/6).$$

He proves that, if $\lambda\to\infty$, $r\to\infty$ so that $|r-\lambda|\lambda^{-1/2}$ remains bounded, we have

$$\lim\,[\lambda^{1-\epsilon}(u_r-v_r)]=0 \quad\text{and}\quad \lim\,[\lambda^{3/2-\epsilon}(u_r-w_r)]=0$$

for any $\epsilon>0$. *Z. W. Birnbaum* (Seattle, Wash.).

Fisz, M. The limiting distribution of a function of two independent random variables and its statistical application. Colloq. Math. **3**, 138–146 (1955).

Let the positive random variable ξ_λ have mean m_λ and finite variance >0, where the parameter λ tends to ∞ in the following. If (a) ξ_λ/m_λ converges to one in probability; (b) ξ_λ is asymptotically normal $N(m_\lambda, \sigma_\lambda)$; (c) ξ_{λ_1} and ξ_{λ_2} are independent and $m_{\lambda_2}/m_{\lambda_1}\to 1$; then $(\xi_{\lambda_2}-\xi_{\lambda_1})(\xi_{\lambda_2}+\xi_{\lambda_1})^{-p}$, $p>0$, is asymptotically normal

$$N[(m_{\lambda_2}-m_{\lambda_1})(m_{\lambda_2}+m_{\lambda_1})^{-p};\ (\sigma^2_{\lambda_1}+\sigma^2_{\lambda_2})^{1/2}(m_{\lambda_1}+m_{\lambda_2})^{-p}].$$

Applications to testing hypotheses are given.

K. L. Chung (Syracuse, N. Y.).

Oderfeld, J. Distribution of the product of rational powers of independent random variables. Zastos. Mat. **1**, 308–320 (1954). (Polish. Russian and English summaries)

Let $X, Y, \cdots$ be independent random variables with known probability distributions, and $\alpha, \beta, \cdots$ rational numbers. Consider the random variable $Z=X^\alpha Y^\beta\cdots$. This paper outlines a numerical procedure for obtaining the cumulative distribution function of Z, based on properties of sequences of numbers $x_i=10^{i/n}$, $i=\cdots, -2, -1, 0, 1, 2, \cdots$, where n is a fixed number of the form $5\cdot 2^k$ with k a non-negative integer. *Z. W. Birnbaum* (Seattle, Wash.).

Greenwood, J. Arthur, and Durand, David. The distribution of length and components of the sum of n random unit vectors. Ann. Math. Statist. **26**, 233–246 (1955).

Let $\xi_1, \cdots, \xi_n$ be independent random angles, each having the density function $g(\xi)$, $0\leqq\xi<2\pi$. Let $V=\sum_i \xi_i \cos\xi_i$, $W=\sum_i \xi_i \sin\xi_i$, $R=(v^2+w^2)^{\frac{1}{2}}$, $P(r, n)=\text{Prob}\ (R\leqq r)$. If g has the circular normal form $\exp\,[k\cos(\xi-\alpha)]/[2\pi I_0(k)]$, it is shown that the likelihood ratio statistic for testing the hypothesis of uniformity ($k=0$) against $k\neq 0$ (α being unspecified) is a function of R; if it is desired to test for uniformity against $k\neq 0$, $\alpha=0$, the statistic is a function of V. The authors tabulate $P(r, n)$ for $r=0.5(0.5)12.0(1.0)17.0$ and $n=6(1)24$, performing a numerical integration of Kluyver's integral solution [Akad. Wetensch. Amsterdam. Proc. **8**, 341–350 (1906)] to Pearson's random-walk problem [Nature **72**, 294, 342 (1905)]. The tabulated values are compared with several approximations. The distribution of V is derived under the hypothesis $\alpha=0$, $k=k_1\neq 0$, affording the power function for testing $k=0$ when α is known to be 0. The distributions of W and (V, W) are calculated for $\alpha=0$, $k=k_1\neq 0$. *T. E. Harris* (Santa Monica, Calif.).

Krishna Iyer, P. V. Random association of points on a lattice. Nature **176**, 40 (1955).

Jiřina, Miloslav. Conditional probabilities on strictly separable σ-algebras. Czechoslovak Math. J. **4**(79), 372–380 (1954). (Russian. English summary)

The conditional probability of a set A relative to a specified σ-algebra **B** of sets is the not uniquely determined density function, defined on the given probability space X, provided by an application of the Radon-Nikodym theorem. It is not always possible to choose this conditional probability in such a way that, for each specified point of X, the conditional probability becomes a completely additive function of A. Under suitable conditions on X, **B**, and the specified σ-algebra **A** of sets A, however, such a choice is possible. The author finds such conditions, generalizing results of the reviewer [Stochastic processes, Wiley, New York, 1953; MR **15**, 445]. For example, sufficient conditions

are that the given probability measure be perfect [as defined in Gnedenko and Kolmogorov, Limit distributions for sums of independent random variables, Gostehizdat, Moscow, 1949; MR **12**, 839; **16**, 52], and that **A** have a countable basis. Other sufficient conditions are that X be a complete separable metric space and that the given probability measure be a measure of the Borel subsets of X. More generally, if X is not separable, it is sufficient if there is a union of a sequence of compact subsets of X, which has probability 1. The author deduces these results from more abstract ones too complicated to state here. *J. L. Doob.*

Kubilyus, I. P. On the distribution of values of additive arithmetic functions. Dokl. Akad. Nauk SSSR (N.S.) **100**, 623–626 (1955). (Russian)

The author extends a theorem of Erdös and Kac on the distribution of additive arithmetic functions [Amer. J. **62**, 738–742 (1940); MR **2**, 42]. The limit normal distribution is replaced by an infinitely divisible one with finite variance under appropriate conditions. *K. L. Chung.*

Dugué, Daniel. L'existence d'une norme est incompatible avec la convergence en probabilité. C. R. Acad. Sci. Paris **240**, 1307 (1955).

If there should exist a norm such that convergence in this norm is equivalent to convergence in probability, then convergence of X_n, $n\geqq 1$, in probability would imply that of $n^{-1}(X_1+\cdots+X_n)$ in probability. Since the latter implication is false it follows that no such norm exists.

K. L. Chung (Syracuse, N. Y.).

Kallianpur, G. On a limit theorem for dependent random variables. Dokl. Akad. Nauk SSSR (N.S.) **101**, 13–16 (1955). (Russian)

Let $X_1, X_2, \cdots$ be a sequence of m-dependent random variables, that is, $(X_1, \cdots, X_r)$ and $(X_s, \cdots, X_n)$ are independent if $s-r>m$. Suppose that $EX_i=0$, $\sup_i E\{X_i^2\}<\infty$, and that $\lim_{n\to\infty} n^{-1}\sum_{p=1}^{n} A_{i+p}=A\neq 0$ uniformly as i varies, where $A_i=E\{X^2{}_{i+m}\}+2\sum_{j=1}^{m} E\{X_{i+m-j}X_{i+m}\}$. Then it is proved that $\sum_1^n X_i/n^{1/2}$ is asymptotically normal, with mean 0 and variance A if, for every $\epsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\int_{|x|>\epsilon n^{1/2}} x^2 dF_j(x)=0,$$

where F_j is the distribution function of X_j. This result generalizes theorems of Hoeffding and Robbins [Duke Math. J. **15**, 773–780 (1948); MR **10**, 200] and of Diananda [Proc. Cambridge Philos. Soc. **49**, 239–246 (1953); **50**, 287–292 (1954); MR **14**, 771; **15**, 635] on m dependent random variables. *J. L. Doob* (Urbana, Ill.).

Kallianpur, Gopinath. On an ergodic property of a certain class of Markov processes. Proc. Amer. Math. Soc. **6**, 159–169 (1955).

The author considers a continuous-parameter Markov process with stationary independent increments, assigning as initial distribution the uniform distribution on $(-\infty, \infty)$ to obtain a stationary process $\{x_t, 0\leqq t<\infty\}$ on a space of infinite measure, and applies the ergodic theorem. Under supplementary hypotheses on the transition probabilities he finds that, if f and g are Lebesgue integrable on $(-\infty, \infty)$, with integrals $\bar{f}$, $\bar{g}\neq 0$, then

$$\lim_{T\to\infty}\int_0^T f(x_t)dt\Big/\int_0^T g(x_t)dt=\bar{f}/\bar{g}$$

with probability 1, for almost all x_0. The reasoning is marred by several lapses, including use of the false statement that, for the usual function-space measure, every measurable set is, apart from a set of measure 0, the union of a sequence of finite-dimensional measurable cylinder sets. The application to Brownian motion yields a result slightly weaker than one due to Derman [Proc. Nat. Acad. Sci. U. S. A. **40**, 1155–1158 (1954); MR **16**, 495] obtained by a different method. (The simple further reasoning necessary to obtain Derman's result is omitted.) Analogous results are obtained for sums of independent random variables [for this case see also Harris and Robbins, ibid. **37**, 860–864 (1953); MR **15**, 140].

J. L. Doob (Urbana, Ill.).

Bailey, Norman T. J. A continuous time treatment of a simple queue using generating functions. J. Roy. Statist. Soc. Ser. B. **16**, 288–291 (1954).

The author solves the difference-differential equations of a queueing process with a single server and Poisson input and output by ingenious manipulation of the probability generating function. These equations had been solved by Ledermann and Reuter [Philos. Trans. Roy. Soc. London. Ser. A. **246**, 321–369 (1954); MR **15**, 625] by another method. *J. Wolfowitz* (Ithaca, N. Y.).

Kawata, Tatsuo. A problem in the theory of queues. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **3**, 122–129 (1955).

The author states: "In this paper, a rather special system will be taken up where there exists a single server, and service times of customers are independently distributed, depending upon the exponential distribution.

"We consider the following situation as to arrivals of customers. When a customer arrives, he may leave, not joining in the queue but giving up receiving service. The probability of departure of an arriver may depend on the number of people waiting before him. In this circumstance we shall discuss the distribution of the number of waiting persons left behind by a departing customer, having received service. The successive arrival time is supposed to be distributed by the Poisson law."

The author considers the process $\{X_n\}$, where X_n is the number in the queue when the service of the nth individual to be served ends, and finds a necessary and sufficient condition for this process to be ergodic. *J. Wolfowitz.*

Shenton, L. R. A semi-infinite random walk with discrete steps. Proc. Cambridge Philos. Soc. **51**, 442–448 (1955).

Consider the successive positions $x_0, x_1, x_2, \cdots$ of a particle performing a random walk over the integers $0, 1, 2, \cdots$ and suppose the one-step transition probabilities,

$$p(j|i)=\Pr(x_{n+1}=j|x_n=i),$$

are such that (1) when $i>0$, $p(j|i)$ is a function of $j-i$ alone, $p(j|i)=0$ for $|j-i|>1$, and $\sum_j p(j|i)=1$; (2) $p(j|0)=0$ for sufficiently large j; and (3) $\sum_j p(j|0)\leqq 1$, the difference $1-\sum_j p(j|0)$ being interpreted as the probability of annihilation. The author represents the n-step transition probabilities

$$p_n(j|i)=\Pr(x_{n+m}=j|x_m=i)$$

by a suitable contour integral. This representation is used to establish expressions for the expected number of steps before annihilation and for the probability that the initial state recur. Various special cases are considered in some detail. *H. P. McKean, Jr.* (Princeton, N. J.).

McKean, Henry P., Jr. Sample functions of stable processes. Ann. of Math. (2) **61**, 564–579 (1955).

Though nowhere stated, this is a study of (separable) processes of homogeneous independent increments with a symmetric stable distribution of a fixed exponent α, $\alpha \varepsilon (0, 2)$. Thus, they are analogues of the Brownian motion process where $\alpha=2$. Several statements about the sample functions $x(t, \omega)$, $t \varepsilon (0, \infty)$, are proved, among which the following. If $\alpha<1$, the Hausdorff dimension of the range of almost all (a.a.) sample functions is α; if $\alpha \geqq 1$ it is 1; both with probability one. The rest of the paper depends on martingale theory. If $\beta>0$ and $\alpha \geqq \beta \leqq 1$ and x_0 is a real number, then $|x(t, \omega)-x_0|^{-\beta}$ is a lower semi-martingale; if $\alpha=1$, then $\log |x(t, \omega)-x_0|$ is an upper semi-martingale. If π is non-negative and Borel measurable and if $\pi(x(t, \omega))$ is a martingale, then π is essentially constant. It follows that the only stationary measures are positive multiples of the Lebesgue measure. The range of a.a. sample functions is nowhere or everywhere dense according as $\alpha<1$ or $\geqq 1$. If $\alpha<1$ and x_0 is given, the probability is zero that for some t either $x(t, \omega)$ or one of the unilateral limits of $x(\cdot, \omega)$ at t is equal to x_0. This result is extended to sets of x_0 of certain capacity zero. [(3.4) is manifestly false: to correct it allow a reversal of the endpoints in (3.4.1).] *K. L. Chung.*

McKean, Henry P., Jr. Hausdorff-Besicovitch dimension of Brownian motion paths. Duke Math. J. **22**, 229–234 (1955).

Let E be a Borel subset of the parameter interval of an n-dimensional Brownian motion, and let $x_\omega(E)$ be the corresponding set on the ωth trajectory. In the following, "dim" means the Hausdorff-Besicovitch dimension. The author proves that, for almost all ω, when $n=1$,

$$\dim x_\omega(E)=\min [2 \dim (E), 1]$$

and, when $n>1$, $\dim x_\omega(E)=2 \dim E$. Lévy [Giorn. Ist. Ital. Attuari **16**, 1–37 (1954); MR **16**, 268] treated the special case in which E is an interval. *J. L. Doob.*

Consael, R. Sur certaines équations fonctionnelles de la théorie des processus markoviens. Assoc. Actuair. Belges. Bull. no. **57**, 63–77 (1954).

L'auteur considère un processus markovien $X(t)$ ayant pour fonction de répartion $F(x, t)$, pour fonction de répartition de passage $F(\xi, \tau; x, t)$. Cette probabilité de passage a une expression, due à W. Feller, qui met en évidence la partie continue et la partie discontinue du processus. Feller a étudié [Math. Ann. **113**, 113–160 (1936)] les équations fonctionnelles, généralisant les équations aux dérivées partielles de Kolmogoroff, que vérifient les fonctions $F(x, t)$ et $F(\xi, \tau; x, t)$. L'auteur retrouve ces équations et complète leur étude par la méthode des fonctions arbitraires. Il introduit une fonction $\Psi(x)$, sommable-$F(x, t)$, admettant une dérivée seconde continue et bornée. Il construit l'équation fonctionelle qui régit l'évolution dans le temps de l'espèrance mathématique $\bar{\Psi}$. Si le processus admet une densité de probabilité ρ et si la fonction de répartition des sauts de $X(t)$ est absolument continue, il est possible de former avec précision un opérateur linéaire A de l'espace L^∞ tel que $d\bar{\Psi}/dt=\overline{A\Psi}$. On peut aussi écrire $d\bar{\Psi}/dt=(\rho, A\Psi)=(A'\rho, \Psi)$, où A' est l'opérateur conjugué de A dans L^1. On en déduit l'équation intégro-différentielle à laquelle satisfait ρ. Lorsque $\Psi=\exp(i\theta x)$, $\bar{\Psi}$ est la fonction caractéristique de $X(t)$.

Quelques cas particuliers sont étudiés, ainsi que le cas des processus à deux composantes $X_1(t)$, $X_2(t)$. On peut faire du X_1 et X_2 certaines hypothèses particulières. L'auteur examine en particulier la forme que prennent les équations générales lorsque $X_1(t)$ admet une dérivée en moyenne quadratique $U(t)$, lorsque $X_2(t)=U(t)$, et enfin lorsque $U(t)$ est aussi dérivable. *J. Bass* (Chaville).

Sevast'yanov, N. B. The problem of search. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1954**, no. 12, 128–131 (1955). (Russian)

The author investigates the smallest distance from the initial point of a search that one of a collection of dispersed targets is sighted. His rather vague definitions and hypotheses lead him to conclusions not involving the search path. *J. L. Doob* (Urbana, Ill.).

Thomson, W. E. The response of a non-linear system to random noise. Proc. Inst. Elec. Engrs. C. **102**, 46–48 (1955).

Wax, Nelson. Signal-to-noise improvement and the statistics of track populations. J. Appl. Phys. **26**, 586–595 (1955).

The problem considered is that of recognizing patterns or tracks formed from a set of observations; for example, deciding whether a set of "blips" on a radar screen indicate the presence of an airplane. It is necessary to prescribe a method for deciding whether a set of signals form a track, and to determine the expected total number of tracks in existence on a screen at a given time. Several schemes are discussed, the most elaborate involving a sequential decision method for confirming or eliminating tracks. Some computations are given. It appears to the reviewer that further development of the statistical aspects of this problem would be of considerable interest. *T. E. Harris.*

Dukes, J. M. C. The effect of severe amplitude limitation on certain types of random signal: a clue to the intelligibility of "infinitely" clipped speech. Proc. Inst. Elec. Engrs. C. **102**, 88–97 (1955).

By "infinite" peak clipping is meant the time quantizing of a signal in which positive amplitudes are replaced by pulses of positive unit amplitude and negative amplitudes are replaced by pulses of negative unit amplitude. It has been found, as a result of experiment, that speech signals when so treated and reproduced are still of high intelligibility, though of poor quality.

For the analysis in the paper, signals are assumed to be generated by a stationary random process. Two classes of signals are examined. In the first class the original signal pulse amplitudes are assumed to be independent. For the most part the paper deals with signals in this category. In the second class the original signal-pulse amplitudes are jointly distributed.

It is shown that with certain signals of common occurrence there is a marked similarity in spectral content before and after clipping. Speech waveforms have certain attributes in common with the idealized signals, and it is therefore concluded that the high intelligibility of "infinitely" clipped speech is a phenomenon to be expected. *S. Kullback.*

***Boonton, R. C., Jr. The analysis of nonlinear control systems with random inputs.** Proceedings of the Symposium on Nonlinear Circuit Analysis, New York, 1953, pp. 369–391. Polytechnic Institute of Brooklyn, New York, 1953. $4.00.

The behaviour of nonlinear elements of control systems is studied in a statistical way. An ensemble of input functions

$x(t)$ of time t is characterized by probability density functions; the problem is to obtain useful information about the statistical properties of the ensemble of output functions $y(t)$. As a typical example a limiter element where $y(t)=f(x(t))$ and $f(x)=c_1 \min(|x|, c_2) \operatorname{sgn} x$ is considered; c_1 and c_2 are positive constants. The output is rewritten as

$$y(t)=Kx(t)+x^*(t)$$

with a constant K (the equivalent gain); K is determined as the minimizing parameter of the integral

$$M=\int_{-\infty}^{\infty}(f(x)-Kx)^2p_1(x)dx$$

with p_1 as the first probability density function of the ensemble $x(t)$. For practical predictions on the behaviour of the nonlinear element, the "distortion" x^* is neglected. This means that the element is replaced by an "equivalent" linear element with $y=Kx$ as relation between input and output. The paper goes into numerical details on K for Gaussian ensembles $x(t)$ and presents also results on feedback control systems with a limiter element. The author indicates how the basic idea of the equivalent linear element can be generalized; he also points out that the investigations should be extended to other typical nonlinear elements, especially to those with backlash. The reviewer believes that a promising start has been made. *H. Bückner.*

Pogorzelski, W. Probability of the safety of a construction. Zastos. Mat. **2**, 46–61 (1954). (Polish. Russian and English summaries)

A system U is considered, consisting of elastic parts U_j, $j=1, \cdots, m$. The maximum σ_j of the absolute value of the stress on U_j is assumed as a function $\sigma_j=F_j(q_1, \cdots, q_n)$ of parameters q_k determined by the geometric and elastic properties of U. If R_j is the strength at failure of U_j and $a_j>1$ a safety coefficient, U is considered safe when $F_j(q_1, \cdots, q_n)<R_j/a_j$ for all $j=1, \cdots, m$. Assuming that $q_1, \cdots, q_n, R_1, \cdots, R_m$ are independent normal random variables, the author writes the multiple integral for the probability that U is safe as a function of the expectations and variances of these random variables and of the a_j. Some generalizations are then carried out under the assumption that the expectations and variances are polynomials in time, with coefficients which themselves are independent normal random variables. An example is discussed in detail. *Z. W. Birnbaum* (Seattle, Wash.).

Pogorzelski, W. Probabilité de la sécurité d'une construction mécanique. Bull. Acad. Polon. Sci. Cl. III. **2**, 363–366 (1954).

The model discussed in the paper reviewed above is generalized by assuming that the parameters q_k are independent normal random variables. *Z. W. Birnbaum.*

Mathematical Statistics

Teichroew, D. Numerical Analysis Research unpublished statistical tables. J. Amer. Statist. Assoc. **50**, 550–556 (1955).

Lieblein, Julius. On moments of order statistics from the Weibull distribution. Ann. Math. Statist. **26**, 330–333 (1955).

The Weibull distribution is described by the c.d.f. $H(x)=0$, for $x\leqq 0$ and $H(x)=1-\exp(-x^m)$, for $x>0$. The author expresses the covariances of order statistics drawn from this distribution in terms of incomplete Beta and Gamma distributions. *B. Epstein* (Stanford, Calif.).

Sundrum, R. M. A further approximation to the distribution of Wilcoxon's statistic in the general case. J. Roy. Statist. Soc. Ser. B. **16**, 255–260 (1954).

The third and fourth moments of Wilcoxon's two-sample test statistic are derived. A short table of values necessary to compute these moments in the normal case is provided. *G. E. Noether* (Boston, Mass.).

Bonferroni, Carlo. I valori mediani in una distribuzione continua. Statistica, Bologna **15**, 3–22 (1955).

Let $x(t)$ and $x_1(t)$ be two Lebesgue measurable functions defined on a finite t-interval; they are said to be permutations of each other if the two t-sets $x(t)\leqq y$ and $x_1(t)\leqq y$ have the same measure. The author introduces and discusses this definition and uses it to study mean values and medians of functions $x(t)$. *E. Lukacs* (Washington, D. C.).

Lukacs, Eugene. Applications of Faà di Bruno's formula in mathematical statistics. Amer. Math. Monthly **62**, 340–348 (1955).

The said formula is about higher derivatives of a function of a function; the application is to a relation between the moments and cumulants of a distribution function, leading to the theorem that a population with finite kth moment is normal if and only if the k-statistic of order $p>1$ is independent of the sample mean. *K. L. Chung.*

Salvemini, Tommaso. Gli indici di connessione nel caso di variabili casuali normali e considerazioni sulla graduatoria tra indici di connessione e indici di concordanza. Statistica, Bologna **15**, 77–90 (1955).

Rowell, J. G. The analysis of a factorial experiment (with confounding) on an electronic calculator. J. Roy. Statist. Soc. Ser. B. **16**, 242–245; discussion 245–246 (1954).

Stuart, Alan. A simple presentation of optimum sampling results. J. Roy. Statist. Soc. Ser. B. **16**, 239–241 (1954).

Grundy, P. M. A method of sampling with probability exactly proportional to size. J. Roy. Statist. Soc. Ser. B. **16**, 236–238 (1954).

Krishna Iyer, P. V., and Singh, Daroga. Problem of distance in sampling. Bull. Inst. Internat. Statist. **23**, part II, 113–118 (1951).

Here, distance means travelling distance. The essential part of the paper is concerned with "free" sampling from a rectangular lattice. The travelling distance is taken to be the column range of the whole sample, plus the sum of all row ranges. For this statistic, the authors compute mean and variance. *G. Elfving* (Helsingfors).

Rao, C. Radhakrishna. On the use and interpretation of distance functions in statistics. Bull. Inst. Internat. Statist. **24**, 2ème livraison, 90–97 (1954).

The paper discusses desirable properties of characteristics measuring the dissimilarity of two populations; such characteristics are the minimax misclassification probability, and Mahalanobis' D^2. Concerning the latter, the following result is proved. Assume that the observable variables x_i have a factor structure $x_i=a_i'\nu+\epsilon_i$, ν being the common-factor

vector; let $E_1(\nu)-E_2(\nu)=\delta$, and cov $\nu=\Gamma$ in both populations. Then $D^2\leqq\delta'\Gamma^{-1}\delta$, no matter how many, and which, of the x_i's are included in the calculation of Mahalanobis' characteristic. *G. Elfving* (Helsingfors).

Ivanovitch, Branislav V. Sur la discrimination des ensembles statistiques. Publ. Inst. Statist. Univ. Paris 3, 207–270 (1954).

Chapter 1 contains a review of the subject and a suggestion that linear discriminant functions be used for simplicity in complicated non-Gaussian cases. Chapter 2 discusses cases where the two possible probability laws of observations are obtained by transforming two given laws. Chapter 3 considers the problem of selecting those two members of a given parametric family which are best separated by a given discriminant function. Chapter 4 is concerned with the problem of estimating the conditional density function of one component of a bivariate random variable given the value of the other, and related problems. *J. Kiefer.*

Laha, R. G. On some problems in canonical correlations. Sankhyā 14, 61–66 (1954).

Consider a $(p+q)$-variate normal population. The author proves two theorems: Theorem 1: The addition of r extra variates to the q-set can never decrease the sum of the squares of the canonical correlations. Theorem 2: The necessary and sufficient condition for the sum of the squares of the canonical correlations to remain the same in both situations (that is, before and after the addition of r extra variates to the q-set) is that all the partial canonical correlations in the population between the p-set and the r-set after eliminating the effect of the q-set, should vanish.
S. Kullback (Washington, D. C.).

Dwass, Meyer. On the asymptotic normality of some statistics used in non-parametric tests. Ann. Math. Statist. 26, 334–339 (1955).

Let $(R_1, R_2, \cdots, R_N)$ be a random vector which takes on each of the $N!$ permutations of $(1, 2, \cdots, N)$ with probability $1/N!$. Let $a_{N1}, \cdots, a_{NN}$ and $b_{N1}, \cdots, b_{NN}$ be real numbers such that

$$\textstyle\sum_i a_{Ni}=\sum_i b_{Ni}=0,\quad \sum_i a^2_{Ni}=N^{-1}\sum_i b^2_{Ni}=1,$$

and put $S_N=\sum_i a_{Ni}b_{NR_i}$. Let $G_N(x)$ denote the proportion of the numbers $b_{N1}, \cdots, b_{NN}$ which are less than x. Theorem 2.1. Suppose that (A) there is a distribution function $G(x)$ such that $\lim_{N\to\infty} G_N(x)=G(x)$ at every point of continuity of $G(x)$; $\int xdG=0$, $\int x^2dG=1$; and either

$$\text{(B)}\qquad \lim_{N\to\infty}\max_{1\leqq i\leqq N}|a_{Ni}|=0,$$

or (B') $G(x)$ is normal. Then S_n has the standard normal limiting distribution as $N\to\infty$. Theorem 3.1 is an extension of Theorem 2.1 to the case where

$$(b_{N1}, \cdots, b_{NN})=(Y_{11}, \cdots, Y_{1N_1}, \cdots, Y_{m1}, \cdots, Y_{mN_m})=Y$$

is a random vector such that the Y_{ij} are mutually independent and $Y_{i1}, Y_{i2}, \cdots$, are identically distributed for each i. Under certain additional assumptions the conditional distribution of S_N given Y converges to the standard normal distribution with probability one. These results extend theorems by Wald and Wolfowitz, the author, and others [for references see Dwass, same Ann. 24, 303–306 (1953); MR 14, 1102]. *W. Hoeffding* (Chapel Hill, N. C.).

Teichroew, D. Empirical power functions for nonparametric two-sample tests for small samples. Ann. Math. Statist. 26, 340–344 (1955).

Tables are given of empirical estimates of the probabilities of rankings of the observations in two samples of sizes m and n from two normal distributions whose means differ by δ times the common standard deviation, for $m=3, 4$, $n=2, 3$, and $\delta=0.00, 0.25, \cdots, 2.50$. The tables suggest that the most powerful rank test against δ small [see M. E. Terry, same Ann. 23, 346–366 (1952); MR 14, 190] retains or almost retains this property throughout the range covered by the tables. *W. Hoeffding.*

Jackson, J. Edward, and Ross, Eleanor L. Extended tables for use with the "G" test for means. J. Amer. Statist. Assoc. 50, 416–433 (1955).

Lord [Biometrika 34, 41–67 (1947); MR 8, 394] provided tables of $u_1=|\bar{X}-\mu|/\{\bar{R}/d_2\sqrt{(nm)}\}$, where $\bar{X}$=sample mean, μ=hypothetical mean, $\bar{R}$=average range of m subgroups of n each, and d_2 is the ratio of the expected value of the sample range to the population standard deviation. The authors provide tables of $G_1=u_1/d_2\sqrt{(nm)}$ to two decimals for $\alpha=.01, .05, .10$ (the levels of significance), $m=1(1)15$, $n=2(1)15$ by converting Lord's u_1 to G_1. Similarly Lord's u_2, $u_2=|\bar{X}_1-\bar{X}_2|d_2/\{\bar{R}\sqrt{(1/nm+1/nm_2)}\}$ is converted to $G_2=\{u_2\sqrt{(1/nm_1+1/nm_2)}\}/d_2$ for the same values of α to two decimals, $n=2(1)15$, m_1, $m_2=1(1)15$. Here $\bar{R}$ is the average range of the combined samples. It is unfortunate in these conversions that G_1 and G_2 are frequently tabulated with only one significant figure. *L. A. Aroian.*

Zítek, F. On certain estimators of standard deviation. Zastos. Mat. 1, 342–353 (1954). (Polish. Russian and English summaries)

An expository paper dealing with estimates for the standard deviation of a normal population. The estimates discussed are the sample standard deviation, mean deviation, Gini's mean difference, the range and some of its generalizations including a statistic recently proposed by Steinhaus. A table of normalizing factors which make these estimates unbiased is given for sample sizes $n=2(1)15$. Numerical illustrations are presented to demonstrate the relative difficulties in computation. Variances are tabulated for $n=2(1)15$ for those estimates for which expressions for the variance were available in literature, and some observations are made on their efficiency. *Z. W. Birnbaum.*

Sadowski, W. On a non-parametric test of comparing dispersions. Zastos. Mat. 2, 161–171 (1955). (Polish. Russian and English summaries)

The author considers k random variables with continuous cumulative distribution functions, assumed to differ only in their variances. To test the hypothesis H that all variances are equal against the unrestricted alternative, he proposes the following procedure: Obtain a sample of the same size n from each of the k populations; reject H if a) one of these k samples contains both the smallest and the largest of all kn sample values and b) the number r of values in this particular sample which are either smaller or greater than all values in the other $(k-1)$ samples is at least equal to a critical value r_0. It is shown that, for fixed k, $\lim_{n\to\infty}$ Prob $\{r\geqq i\}=[i-1-(i-2)/k]/k^{i-1}$. Exact values of Prob $\{r\geqq i\}$ are tabulated for $k=2, 3, 4$, and $n=2, 3, \cdots$ up to n such that the limit distribution yields a good approximation. *Z. W. Birnbaum* (Seattle, Wash.).

Douglas, J. B. Fitting the Neyman Type A (two parameter) contagious distribution. Biometrics 11, 149–173 (1955).

The author presents a simplified maximum-likelihood procedure for the estimation of the two parameters μ and v in the Neyman Type A contagious distribution. The case where the zero class is unknown is briefly discussed. Given $\lambda=\mu e^{-v}$, define

$$\mu_x'=e^{-\lambda}\sum_{r=0}^{\infty} r^x\frac{\lambda^r}{r!},\quad p_x=\mu'_{x+1}/\mu_x',\quad q_x=p_x\ (p_{x+1}-p_x).$$

The author provides tables of p_x, and q_x, $x=0(1)19$, $\lambda=.000(.001).03(.01).3(.1)3$, at least to two decimals and in some cases as many as five. *L. A. Aroian.*

Czechowski, T., Fisz, M., Sadowski, W., and Zasępa, R. On determining the safety factor. Zastos. Mat. 2, 190–198 (1955). (Polish. Russian and English summaries)

The strength R of a structural material is assumed to be a normal random variable with expectation R_0 and variance σ^2. If the specifications require that the greatest stress must not exceed R_0/Γ, with $\Gamma>1$ such that Prob $\{R<R_0/\Gamma\}=\alpha$ for given α, then the resulting value $\Gamma=R_0/(R_0+l_\alpha\sigma)$ is called the "safety factor". In practical situations R_0 and σ are not known, and the authors propose that Γ be estimated by $\gamma=\bar{r}/(\bar{r}+l_\alpha s)$, where $\bar{r}$ is the sample mean and s^2 the sample variance of n observations of R. The problem considered is to determine n so that Prob $\{|\gamma-\Gamma|<\epsilon\}=\beta$, for given $\epsilon>0$, $0<\beta<1$. *Z. W. Birnbaum.*

Girshick, M. A., Rubin, H., and Sitgreaves, R. Estimates of bounded relative error in particle counting. Ann. Math. Statist. 26, 276–285 (1955).

The problem of estimating the average number of events occurring per unit of a continuous variable is considered when the underlying distribution of events is assumed to be Poisson. An estimate l of the parameter λ is said to be an estimate of bounded relative error if for a specified confidence coefficient α, l does not differ from λ by more than 100γ per cent of λ, where α and γ do not depend on λ. Since the usual estimate based on counting the number of events occurring in independent sub-areas of the sample space is not an estimate which is of bounded relative error, other approaches are considered. A sampling procedure is proposed in which the continuous variable is observed until a fixed number M of events occur. Such a procedure enables the introduction of an estimate which is of bounded relative error. Some tabulated results relating α, γ, and M are given. Modifications of the procedure which are of a sequential nature are described. *M. Muller* (Ithaca, N. Y.).

Rajski, C. Comparing general populations on the basis of Bayes' rule. Zastos. Mat. 1, 330–341 (1954). (Polish. Russian and English summaries)

Let Π_1, Π_2 be two dichotomized populations, w_1, w_2 fractions of individuals of one kind in these populations, n_1, n_2 the sizes of samples obtained from Π_1, Π_2, and r_1, r_2 the numbers of individuals of the specified kind contained in these samples. Making the assumption that w_1, w_2 are themselves independent random variables, each uniformly distributed, and using Bayes' theorem, the author formulates procedures for "verifying" various hypotheses on w_1, w_2, that is for computing the conditional probabilities that, given n_1, n_2, r_1, r_2, the fractions w_1, w_2 satisfy these hypotheses. *Z. W. Birnbaum* (Seattle, Wash.).

Lange, O. Statistical estimation of parameters in Markov processes. Colloq. Math. 3, 147–160 (1955).

"In the present paper we shall consider the statistical estimation of parameters in the following elementary Markov processes: the simple Poisson process, the Gaussian process with stationary independent increments which is usually called the Brownian motion process, the linear "birth process" and the linear "death process". Finally we shall consider the case of estimating transition probabilities in simple Markov chains." (Extract from the paper.) *J. Wolfowitz* (Ithaca, N. Y.).

Bartlett, M. S. A note on the multiplying factors for various χ^2 approximations. J. Roy. Statist. Soc. Ser. B. 16, 296–298 (1954).

A list is given of various approximate tests proposed by the author over the past several years and based on the asymptotic approximations for likelihood ratios, but with adjusted multiplying factors. A correction is noted to the multiplying factor suggested in the paper by Bartlett and Rajalakshman [same J. 15, 107–124 (1953); MR 15, 333]. *D. G. Chapman* (Seattle, Wash.).

Ihm, Peter. Ein Kriterium für zwei Typen zweidimensionaler Normalverteilungen. Mitteilungsbl. Math. Statist. 7, 46–52 (1955).

Given a bivariate normal distribution, a test of the hypothesis $\sigma_1^2=\sigma_2^2$, $\rho=0$ based on the likelihood ratio is derived and discussed. *G. E. Noether* (Boston, Mass.).

Anderson, T. W., and Darling, D. A. A test of goodness of fit. J. Amer. Statist. Assoc. 49, 765–769 (1954).

The authors discuss the test of goodness of fit based on the statistic

$$W_n^2=n\int_{-\infty}^{+\infty}[F_n(x)-F(x)]^2\psi[F(x)]\,dF(x)$$

with the weight function $\psi(u)=1/u(1-u)$. They show that this statistic can be written in a form which lends itself to numerical computation, and give its asymptotic 10%, 5%, and 1% significance points. Empirical evidence is quoted as suggesting that these asymptotic values are safe already for $n\geqq 40$. A numerical illustration is presented. *Z. W. Birnbaum* (Seattle, Wash.).

Sakaguchi, Minoru. On minimax tests of hypotheses. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. 3, 130–139 (1955).

The tool of using Bayes strategies or sequences of Bayes strategies to find minimax tests [see, e.g., Theorem 8.3 of E. L. Lehmann, Ann. Math. Statist. 21, 1–26 (1950); MR 11, 528] is applied to some Gaussian examples. *J. Kiefer* (Ithaca, N. Y.).

Hodges, J. L., Jr., and Lehmann, E. L. Testing the approximate validity of statistical hypotheses. J. Roy. Statist. Soc. Ser. B. 16, 261–268 (1954).

The authors discuss the difference between statistical significance and "material" significance in hypothesis testing, and apply this to several common problems. A test of the (not necessarily simple) hypothesis H is materially significant on the level α, say, if α is the maximum probability of rejecting H when the true distribution is in some prescribed neighborhood of H. *J. Wolfowitz.*

Hannan, E. J. Exact tests for serial correlation. Biometrika 42, 133–142 (1955).

It is shown that Ogawara's exact test for serial correlation [Ann. Math. Statist. 22, 115–118 (1951); MR 12, 726] is asymptotically fully efficient if the parent scheme is a first-order autoregression. The author also extends this test to give an exact test of the independence of residuals from a regression equation, and of given values of the regression coefficients.

P. Whittle (Wellington).

Theil, H. Estimation of parameters of econometric models. Bull. Inst. Internat. Statist. 24, 2ème livraison, 122–129 (1954).

A formal treatment of parameter estimation in a linear system of jointly dependent variables [cf. W. Hood and T. Koopmans, eds., Studies in econometric method, Wiley, New York, 1953; MR 15, 812]. Writing the relations in the "normalised" form

$$(*)\quad y_1(t)=\sum_2^m\alpha_r y_r(t)+\sum_1^n\beta_s x_s(t)+\gamma+u(t)\quad(t=1,\cdots,T),$$

a class $G(k)$ of estimates is defined which for $k=0$, $k=1+\theta$ and $k=1$, respectively, yields estimates of α_r, β_s according to (A) least squares regression, (B; subject to a specified interpretation of θ) the limited-information method [see loc. cit., p. 166], and (C) the generalised minimum-variance method of A. C. Aitken [Proc. Roy. Soc. Edinburgh 55, 42–48 (1935)] applied after transforming (*) by using the $x_s(t)$ as instrumental variables in the sense of O. Reiersöl [Ark. Mat. Astr. Fys. 32A, no. 4 (1945); MR 7, 317]. The following theorem in favour of (C) is announced and briefly discussed: On a general specification of (*) a sufficient condition for the $G(k)$ estimates to be consistent is that plim $(k-1)=0$, $T\to\infty$; the asymptotic covariance matrix of the estimates is presented, subject to the condition plim $(k-1)\sqrt{T}=0$. Many disturbing errata, notably: Read $\leqq 1$ for $\sum_1$ at the end of p. 127, $H\cdot$ for $H=$ in (2.6), and $\sigma^2\cdot\text{plim}\,G^{-1}$ to the right in (2.7). The reviewer finds the approach formalistic, for the representation (*) is less innocent than a "normalisation", inasmuch as (*) should be treated by method (A) only if it is legitimate to predict $y_1(t)$ in terms of the right-hand variables, whereas in situations (B) and (C) such prediction has no optimum property or may sometimes not even be logically permissible.

H. Wold (Uppsala).

Bhattacharyya, A. The problem of regression in a statistical population admitting location parameters. Bull. Inst. Internat. Statist. 23, part II, 29–54 (1951).

Let f be a given density function with respect to Lebesgue measure on R^n and let $Y_1,\cdots,Y_n$ be random variables whose joint density function is

$$\sigma^{-n}f([y_1-\lambda_1]/\sigma,\cdots,[y_n-\lambda_n]/\sigma)$$

where σ (>0) and the λ_i (real) are unknown parameters subject to the restrictions $\lambda_i=\sum_{j=1}^p\beta_j x_{ji}$, the x_{ji} being given. Under suitable regularity conditions, the author obtains expressions for estimators of σ and the β_j (based on $Y_1,\cdots,Y_n$) which have minimum expected squared error among the class of estimators satisfying an appropriate cogredience condition. Some other problems, including that of interval estimation in the above case, are discussed.

J. Kiefer.

★**Dommanget, J. Etude des droites caractérisant soit une liaison fonctionnelle, soit une dépendance statistique entre deux variables.** L'exploitation des données empiriques. Publ. Sci. Tech. Ministère de l'Air, Paris, Notes Tech. no. 52, pp. 29–39 (1955). 850 francs.

An elementary consideration of the rationale of least-square linear regression. *P. Whittle* (Wellington).

Wishart, John. Multivariate analysis. Appl. Statist. 4, 103–116 (1955).

Expository paper.

Hartley, H. O. Some recent developments in analysis of variance. Comm. Pure Appl. Math. 8, 47–72 (1955).

The paper concentrates on (1) multiple decisions and comparisons, (2) substitute measures of dispersion. It is partly a summary of previous work, with extensive references. Specific contributions are the following. (1) A sequential F-test for multiple decisions on significance. Let $s^2; s_1^2,\cdots,s_k^2$ be independent mean squares, obtained in an analysis of variance, and 'due' to error and various series of treatments, respectively. If, for example, the latter mean squares have the same degree of freedom, the procedure consists in comparing the ratios s_i^2/s^2, taken in descending order of magnitude, with appropriate percentage points, until a non-significant ratio is reached. Inequalities are derived for the probability of wrongly returning at least one s_i^2 as significant, and for the power with respect to a single effect. (2) A study of substitute measures of dispersion, particularly of their correlation with the standard measure. The results support the use of mean range in many practical situations. *G. Elfving* (Helsinki).

Wise, J. The autocorrelation function and the spectral density function. Biometrika 42, 151–159 (1955).

The author considers some approximate results obtained by the reviewer [Thesis, Uppsala, 1951; MR 12, 726] concerning the representation and properties of the expected covariance matrix V of a finite sample from a stationary time series, and derives the analogous exact results which would be valid if V were circulant. For the non-circulant case, some formal calculations are carried out for the semi-infinite sample, and an exact expression quoted for V^{-1} for a finite sample from an autoregressive series. *P. Whittle.*

Bartlett, M. S., and Medhi, J. On the efficiency of procedures for smoothing periodograms from time series with continuous spectra. Biometrika 42, 143–150 (1955).

The authors investigate the optimum method of periodogram smoothing in a manner similar to Grenander's [Ark. Mat. 1, 503–531 (1951); MR 14, 187] but differing in that the estimate of every spectral ordinate is required to be asymptotically unbiased. A generalised form of Grenander's uncertainty relation is deduced, and efficiencies of various smoothing methods compared. An optimum smoothing formula is derived, which in the case of narrow-band smoothing yields a simple modification of Daniell's formula.

P. Whittle (Wellington).

Mathematical Biology

Rushton, S., and Mautner, A. J. The deterministic model of a simple epidemic for more than one community. Biometrika 42, 126–132 (1955).

A simple epidemic is one in which it is assumed that infection spreads only by contact between individuals and

that none of the infected individuals is removed from circulation by death, recovery or isolation. A deterministic model is set up for the spread of such a simple epidemic in case the whole community is divided into subgroups with different rates of infection within and between subgroups. The differential equations of this model are solved for the case that all subgroups are of the same size and the within infection rates are the same for all subgroups; also it is assumed that the rate of infection between subgroups is the same for all pairs. Particular attention is paid to the important special situation where one infected person is initially introduced into one subgroup, all others being initially uninfected. Some numerical results are exhibited for this case. *D. G. Chapman* (Seattle, Wash.).

Karlsson, Georg. Note on the spread of a state in small social groups. Bull. Math. Biophys. 17, 1–5 (1955).

Let $b_{jk}(t)$ represent the proportion of the tth discrete time interval spent by individual j in behavior k; a_{ij} the influence of individual j on the behavior of individual i, with the assumption that $B(t+1)=AB(t)$, where A and $B(t)$ are the influence and behavior matrices. The matrix A is a stochastic matrix. One defines "transient", "periodic" and "ergodic" behavior, "closed groups", etc. in a natural way. Standard theorems on stochastic matrices are then applied and interpreted. *A. S. Householder* (Oak Ridge, Tenn.).

Hearon, John Z. Note on the theory of mass behavior. Bull. Math. Biophys. 17, 7–13 (1955).

Rapoport [same Bull. 14, 159–169 (1952); MR 13, 963] obtained the equation $dF/dt=(1-F)[x(t)+bF]$ in the study of the propagation of an act in a large population. In this paper the author points out that the equation is a Riccati equation and deduces some properties of the solution from this. *A. S. Householder* (Oak Ridge, Tenn.).

Ludwig, Wilhelm. Probleme und Aufgaben der Biomathematik. Studium Gen. 6, 637–646 (1953).

Hofstaetter, Peter R. Psychologie und Mathematik. Studium Gen. 6, 652–662 (1953).

de Rudder, B. Mathematik in der Medizin. Studium Gen. 6, 647–651 (1953).

TOPOLOGY

Borsuk, K. What is topology? Wiadom. Mat. (2) 1, 65–74 (1955). (Polish)

Elementary expository paper.

Inokuma, Takeshi. On a characteristic property of completely normal spaces. Proc. Japan Acad. 31, 56–59 (1955).

It is shown that a space is completely normal if (for $n=2$) and only if (for all n) whenever sets $A_1, \cdots, A_n$ are given, there is a closed covering $H_1, \cdots, H_n$ with

$$H_i \cap H_j \cap (A_i^- \cup A_j^-) = A_i^- \cap A_j^-$$

for all i, j. When there are infinitely many A_i and H is fully and completely normal, such H_i can be found when the A_i form a locally finite subset. *R. Arens.*

Morita, Kiiti. On spaces having the weak topology with respect to closed coverings. II. Proc. Japan Acad. 30, 711–717 (1954).

[For part I see same Proc. 29, 537–543 (1953); MR 15, 977.] Let X be a space having the weak topology with respect to a closed covering $\{A_\alpha\}$ (i.e. the union of any subcollection $\{A_\beta\}$ is closed in X, and U is open in X if and only if $U \cap A_\alpha$ is open for each A_α). The author proves: X is paracompact (countably paracompact) and normal if and only if each A_α is paracompact (countably paracompact) and normal. Reviewer's remark: The lengthy proof given here can be replaced by a simple transfinite induction starting from a result of E. Michael [Bull. Amer. Math. Soc. 59, 180 (1953)]. *J. Dugundji* (Los Angeles, Calif.).

Ivanova, V. M. On the theory of spaces of subsets. Dokl. Akad. Nauk SSSR (N.S.) 101, 601–603 (1955). (Russian)

Let E be a topological space and let $\mathcal{S}(E)$ be the set of all non-void closed subsets of E. For an open subset G of E, let $\mathfrak{U}_1(G)$ be the set of all $F \in \mathcal{S}(E)$ such that $F \subset G$, and let $\mathfrak{U}_2(G)$ be the set of all $F \in \mathcal{S}(E)$ such that $F \cap G \neq 0$. $\mathcal{S}(E)$ is given the topology such that the family of all sets $\mathfrak{U}_1(G)$ and $\mathfrak{U}_2(G)$ forms a sub-basis for open sets. Let τ be a limit ordinal, and let J_τ be the space of all ordinals $<\tau$, with the order topology. Then $\mathcal{S}(J_\tau)$ is non-normal. *E. Hewitt.*

Bagley, Robert, and Ellis, David. On the topolattice and permutation group of an infinite set. Math. Japon. 3, 63–70 (1954).

Let S be an infinite set, and let Λ be the set of all T_1-topologies on S. For $\alpha, \beta \in \Lambda$, the authors write $\alpha \leqq \beta$ if every set open with respect to β is also open with respect to α. It is known and obvious that Λ is a complete lattice with a greatest and a least element. Let $\bar{\Lambda}$ be the dual lattice to Λ ($\alpha \leqq \beta$ in $\bar{\Lambda}$ if and only if $\beta \leqq \alpha$ in Λ). The authors construct a certain class of isotone mappings of $\Lambda \otimes \bar{\Lambda}$ onto the 2-element lattice $\{0, 1\}$ and show that these mappings are in 1-to-1 correspondence with the class of all permutations of S. *E. Hewitt* (Princeton, N. J.).

Banaschewski, Bernhard. Abstufungen des Kompaktheitsbegriffes. Arch. Math. 6, 320–329 (1955).

The author discusses several quantitative variations of the different formulations of compactness in terms of open coverings, of cluster points of filters, and of limits of ultrafilters. The discussion turns about the notion of upper characteristic filter of a set A; this is defined to be the filter of all subsets of A whose complements are of cardinal number less than that of A. *M. M. Day* (Urbana, Ill.).

Sierpinski, W. Sur les espaces métriques séparables contenant un nombre fini de points d'accumulation. Matematiche, Catania 9, 122–125 (1954).

Separable metric spaces without points of accumulation are countable and discrete. The structure of spaces with but finitely many points of accumulation is resolved in this note. The author constructs, for each natural number s, $s+2$ sets, $E_1, \cdots, E_{s+2}$, of real numbers each with precisely s points of accumulation. These sets have the following property. A separable metric space E with precisely s points of accumulation is homeomorphic to exactly one of the sets $E_1, \cdots, E_{s+2}$. *M. E. Shanks* (West Lafayette, Ind.).

Kurepa, G. Some remarks on abstract spaces. Bull. Internat. Acad. Yougoslave. Cl. Sci. Math. Phys. Tech. (N.S.) **12**, 43–50 (1954).

Can an isolated set be separated simultaneously by means of pairwise disjoint open sets? If the space is either metric or totally ordered, the answer is "yes". [Surely both of these elementary facts are well known.] For more general spaces, e.g. for uniform spaces, the question is open. [The answer for uniform spaces is "no". There exists a separable normal Hausdorff space which contains an uncountable isolated set. See the example in the reviewer's paper, Bull. Amer. Math. Soc. **43**, 671–677 (1937). For normal semi-metric or normal Moore spaces the question is open.]

If P is a space and sP is the supremum of cardinals of families of pairwise disjoint open subsets of P, then is the cardinality of one such family sP? If the space is either metric or totally ordered with sP not inaccessible, the answer is "yes". In these two situations, the cardinality of no isolated set exceeds sP. [This is not true for semi-metric or Moore spaces. See M. E. Estill, Duke Math. J. **18**, 623–629 (1951); MR **13**, 148.]

If S is a metric set and $d_a S$ is the largest real number such that every subset of S whose cardinality exceeds the cardinality of its complement (relative to S) has diameter not less than $d_a S$, then $d_a S$ is called the asymptotic diameter of S. If E_0 is the set of all subsets X of a metric space E of asymptotic diameter zero ($d_a X=0$), then E_0 is a pseudometric space which in the standard fashion yields a complete metric space E^* in which E is isometrically and densely embedded. *F. B. Jones* (Chapel Hill, N. C.).

Kurepa, Đuro. Remarks on abstract spaces. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **296**, 95–103 (1953). (Serbo-Croatian)

Serbo-Croatian version of the paper reviewed above.

Burgess, C. E. Certain types of homogeneous continua. Proc. Amer. Math. Soc. **6**, 348–350 (1955).

Having previously shown [same Proc. **5**, 136–143 (1954); MR **15**, 814] that if an n-homogeneous ($n>1$) plane continuum is bounded it is a simple closed curve, the author now shows that if it is unbounded it is either the entire plane or a simple open curve (i.e., the homeomorph of a straight line). To do so he points out that every nearly n-homogeneous ($n>1$) plane continuum (whether bounded or not) is either locally connected or indecomposable. More specifically, if every proper subcontinuum of a compact metric continuum M (not necessarily planar) is nearly 1-homogeneous (=homogeneous) then M is hereditarily indecomposable. Finally, a simple closed curve is characterized as a decomposable compact metric continuum M such that if H and K are non-degenerate proper subcontinua of M there is a homeomorphism of M onto M which carries H onto K. *F. B. Jones* (Chapel Hill, N. C.).

Barrett, Lida K. Regular curves and regular points of finite order. Duke Math. J. **22**, 295–304 (1955).

J. R. Kline has raised the question as to whether or not there exists, for a given integer n, a locally compact continuous curve S such that each pair of points of S are the end points of n independent arcs of S, but no two points of S are the end points of $(n+1)$ independent arcs of S. The first part of this paper proves that no such space exists for $n>2$. The same result for $n=3$ and $n=4$ has previously been obtained by J. H. Kusner [C. R. Soc. Sci. Lett. Varsovie. Cl. III. R. **25**, 71–92 (1933)].

The final part of the paper answers two questions raised by W. L. Ayres concerning continuous curves containing points of only two orders. Urysohn [Verh. Akad. Wetensch. Amsterdam **13**, no. 4 (1927), pp. 109–115] has constructed examples of curves containing points of only two orders, namely, for any integer $n>2$, a curve containing points of orders n and $(2n-2)$ only. G. T. Whyburn [Bull. Amer. Math. Soc. **35**, 218–224 (1929)] has shown that if a continuous curve contains points of only two orders, m and n, then $m\geqq 2n-2$. The present paper proves that for any pair of integers m and n such that $m\geqq 2n-2$ there exists a continuous curve composed entirely of points of orders m and n. Ayres conjectured [Trans. Amer. Math. Soc. **33**, 252–262 (1931)] that if $m>n>2$, then (1) the points of order m must be countable and (2) for some integer k, $m=k(n-1)$. It is shown by several examples that neither of these conjectures is true. *D. W. Hall* (College Park, Md.).

Dolcher, Mario. Geometria delle trasformazioni continue. Un teorema sulle trasformazioni di varietà semplici n-dimensionali. Ann. Univ. Ferrara. Sez. VII. (N.S.) **3**, 11–16 (1954).

In oriented euclidean n-space let $\bar{C}$ be a simple variety (topologically, an oriented n-dimensional polyhedron), let Φ be a continuous transformation on $\bar{C}$, and let γ' be the image under Φ of the oriented boundary of $\bar{C}$. If A' is a component of the complement of γ' on which the topological index of every point with respect to γ' is not zero it is shown that there exists a component of $\bar{C}-\Phi^{-1}(\gamma')$ whose image under Φ contains A'. *P. V. Reichelderfer.*

Deheuvels, René. Topologie d'une fonctionnelle. Ann. of Math. (2) **61**, 13–72 (1955).

This paper brings a detailed account of the results which the author announced in earlier notes [C. R. Acad. Sci. Paris **235**, 778–780, 858–860, 1270–1272 (1952); **238**, 1186–1188 (1954); MR **14**, 492; **15**, 890]. The author here generalizes, clarifies and refines the theory of critical values and points, of a numerical function φ on a space X, which was initiated and developed by M. Morse. To summarize the author's constructions, let us suppose that the singular cohomology theory is chosen as tool with which the local behavior of φ on X is to be studied. The singular cochains of X with coefficients Ω, say A, can then be filtered by sets A^p where p runs over the ordered set S: $-\omega<-\infty<R<\infty<\omega$, by setting A^p equal to those cochains which are zero on the set of singular simplexes of X for which $\phi<p$. Then $A^{-\omega}$ is the vacuous set, and $A^\omega=A$. Let $E(p,q)=H(A^p/A^q)$ $(p\leqq q)$. Thus if φ is upper semicontinuous

$$E(p,q)=H([\varphi<q],[\varphi<p]).$$

The ordered pairs (p,q) form a lattice under the partial order $(p,q)\leqq(p',q')$; $p\leqq p'$; $q\leqq q'$. Clearly if $(p,q)\leqq(p',q')$ the inclusion $E(p,q)\leftarrow E(p',q')$ is a well-defined homomorphism. Similarly the boundary operator δ: $E(p,q)\rightarrow E(q,r)$ can be defined in the usual fashion if $p\leqq q\leqq r$, and these homomorphisms satisfy the usual commutativity theorems as well as the exact sequence of a triple. The lattice of pairs (p,q) together with this assignment $(p,q)\rightarrow E(p,q)$ the author denotes by $\mathfrak{L}(\varphi)$ and refers to as the lattice of relative homologies associated with φ. The theory of critical values of φ, has, from this point of view, the aim to relate the local groups at various levels (i.e. $E(p,q)$ in the vicinity of the diagonal $p=q$) with the group $E(-\omega,\omega)=H(X)$. In Morse's scheme of things the local groups are essentially defined as the groups $\lim_{0<\epsilon\to 0} E(p,p^{+\epsilon})=E(p,p^+)$. So that critical values of φ would be those r for which $E(r,r^+)=0$. The author

observes that this definition leads to a proper theory only under certain conditions on φ, X, and H. He therefore refines the concept of a critical value in the following fashion: Let $E(p, q, r, s)$ be the image of $E(p, r) \dot{\leftarrow} E(q, s)$. Now the author describes the local behavior of φ at level r by the groups

$$E(r, r, r^+, r^+) = \lim_{\substack{0<\eta\to 0 \\ 0<\epsilon\to 0}} E(r, r, r+\epsilon, r+\eta).$$

Then the direct sum $\Gamma = \sum_{r \epsilon S} E(r, r, r^+, r^+)$ is defined as the critical ring of $\mathfrak{L}(\phi)$, and its relation to $H(X)$ is described by the following version of the Morse inequalities (if the coefficients are a field): There exists a differential operator d on Γ, such that the derived group of Γ with respect to d is (unnaturally) isomorphic to $H(X)$. This proposition easily translates into the classical Morse inequalities under appropriate finiteness conditions. The definition of Γ involves the consideration of the limiting values of the relative homologies of four sets. As in Leray's spectral sequence, this number 4 is in a sense universal, for as the author shows, no additional information would be obtained by studying more than 4 relative sets at each level. Using the singular cohomology, the author has therefore developed a theory of critical values which is universally applicable. The special feature of this homology theory, as well as of the filtering used, is that it is automatically left-continuous. Just as the degree to which right continuity fails is measured by Γ, the corresponding left discontinuities could be measured by an appropriate inverse-limit of groups. However, as the inverse limit in general does not preserve exactness, such a theory would be complete only under certain additional conditions on X and φ. As mentioned earlier, the author develops this theory in great generality, starting with a discussion of partially ordered sets, their limits and inverse limits, he then discussed filtrations on various structures.

In Chapter III the relative homology lattice $\mathfrak{L}$ is introduced quite abstractly by axioms. Chapter IV contains the constructions which are at the basis of Morse's inequalities. Specially noteworthy here is the introduction of the ring Δ, the dicritical ring of $\mathfrak{L}(\varphi)$, which is later identified as the graded ring of the critical ring Γ. Chapter V is devoted in the main to the theory of critical points of φ. Here in distinction with the above theory of critical values, some additional compactness condition is seen to be necessary for the existence of critical points at each critical level.

R. Bott (Princeton, N. J.).

Mardešić, Sibe. Sur un problème de M. Borsuk concernant l'homologie de l'espace fonctionnel $S_m{}^X$. C. R. Acad. Sci. Paris **240**, 2287–2288 (1955).

In this note, the author solves a problem raised by K. Borsuk [Fund. Math. **39**, 25–37 (1953); MR **15**, 51]. The main theorem is stated as follows: Let X be a compact space with $\dim X \leq k$, $k>0$, and P denote the group of real numbers mod 1. If $H_k(X; P)=0$, then $H_{m-k}(S_m{}^X; G)=0$ for every coefficient group G. Here $S_m{}^X$ denotes the space of the continuous maps of X into the m-sphere S_m with the compact-open topology. Following Borsuk, the author uses the homology theory based on the Vietoris convergent cycles with compact carriers. *S. T. Hu* (Athens, Ga.).

Berikašvili, N. A. On duality theorems for arbitrary sets. Soobšč. Akad. Nauk Gruzin. SSR **15**, 407–414 (1954). (Russian)

Given a subset M of a space L, a covering of M by open sets from L is called outer, and a covering of M by relatively open subsets of M inner. With the system of outer (inner) coverings there may be associated in the usual way Čech and Vietoris homology theories. It is shown that, if the space L is separable metric, there is no difference between the outer and inner homology theories, and furthermore in this case the Čech and Vietoris versions coincide. (Actually, this last result holds for paracompact spaces.) The proof of the first result is based on Kaplan's theorem to the effect that in a separable metric space any open covering admits a countable star-finite refinement [Trans. Amer. Math. Soc. **62**, 248–271 (1947); MR **8**, 456]. $V_1{}^k(M, X)$ will denote the homology group of dimension k of M thus obtained over the coefficient group X. If X is discrete/compact, $V_1{}^k(M, X)$ is also discrete/compact. Furthermore, the compact subsets of M give rise to a direct system of homology groups. If the coefficient group X is discrete, the limit group will be denoted by $V_2{}^k(M, X)$. If X is compact, the direct limit can be topologized by a procedure due to Čogošvili [Mat. Sb. N.S. **28(70)**, 89–118 (1951); MR **12**, 846] so as to give a precompact group. The compact completion will be denoted by $\bar{V}_2{}^k c(M, X)$. In the nerve of each covering of M a compact subset of M gives rise to a subcomplex which is finite if one restricts oneself to star-finite coverings. The system of compact subsets gives rise to a direct system of finite subcomplexes in the nerve of a given covering. Taking the system of all coverings of M, one gets an "inverse direct system of finite complexes". Taking for a compact coefficient group Y first the compactified Čogošvili limit, and then the (ordinary) inverse limit of the corresponding system of homology groups, one gets a third kind of homology groups $E^k(M, Y)$ for M. One has the following duality theorems: If F and G are complementary subsets of the n-sphere, X a discrete abelian group and Y its character group, then $V_1{}^k(F, X)$ is dual to $\bar{V}_2{}^r c(G, Y)$, and $V_2{}^k(F, X)$ is dual to $E^r(G, Y)$, $k+r=n-1$. The first result is due to Alexandroff and Čogošvili, the second to the author.

W. T. van Est (Utrecht).

Inoue, Yoshiro. Some remarks on abhomotopy groups. Proc. Japan Acad. **31**, 60–65 (1955).

The structure theorems of the abhomotopy groups [Hu, Ann. of Math. (2) **48**, 717–734 (1947); MR **9**, 197] and the relative abhomotopy groups [Inoue, Proc. Japan Acad. **30**, 841–845 (1954); MR **16**, 948] are re-established by means of some fiber spaces (in the sense of J.-P. Serre) with natural cross-sections. As an application, a counter example is given to the statement (4.1) in a paper of the reviewer [Portugal. Math. **5**, 219–231 (1946); MR **8**, 481]; the original proof of this statement contains a mistake. *S. T. Hu.*

Hilton, P. J. On the homotopy groups of unions of spaces. Comment. Math. Helv. **29**, 59–92 (1955).

Let X be a CW-complex, and X^r its r-dimensional skeleton. J. H. C. Whitehead has defined $\Gamma_r(X)$ to be the kernel of the natural homomorphism j_r: $\pi_r(X)^r \to \pi_r(X^r, X^{r-1})$, and shown that there is an exact sequence

$$\cdots \to \Gamma_r(X) \to \pi_r(X) \to H_r(X) \to \Gamma_{r-1}(X) \to \cdots$$
$$\to \pi_3(X) \to H_3(X) \to 0$$

when X is a 2-connected space. Further, if X is $(p-1)$-connected, $p \geq 3$, then $\pi_p(X) \otimes Z_2 \simeq \Gamma_{p+1}(X)$.

Now if $\alpha \in \pi_p(X)$, let $\bar{\alpha}$ denote the corresponding element of $\Gamma_{p+1}(X)$. Moreover, let λ: $\Gamma_r(X) \to \pi_r(X)$ denote the natural homomorphism. Assume that X and Y are CW-complexes, $X \times Y$ is their Cartesian product, and $X \vee Y$ their union with a point in common. The main theorem of

the author then says that if X is $p-1$ connected, and Y is $q-1$ connected, where $p, q \geq 3$, then there is an exact sequence

$$0 \to G \to \pi_{p+q+1}(X \times Y, X \vee Y) \to \mathrm{Tor}\,(H_p(X), H_q(Y)) \to 0,$$

where G is the group obtained from

$$\pi_p(X) \otimes \pi_{q+1}(Y) + \pi_{p+1}(X) \otimes \pi_q(Y)$$

by identifying $\alpha \otimes \lambda\bar{\beta}$ with $\lambda\bar{\alpha} \otimes \beta$. There are many other theorems giving properties of the homotopy of $(X \times Y, X \vee Y)$.

J. C. Moore (Princeton, N. J.).

Ebersold, Johannes M. Über die Rolle des Whiteheadschen Homotopieproduktes für die Homologietheorie. Compositio Math. 12, 97–133 (1954).

Let K be a finite complex and $n>2$. Denote by $W_{n+1} \subset \pi_{n-1}(K^{n-1})$ the subgroup generated by the collection of all Whitehead products $\alpha_l \circ \alpha_{n-l}$ where $\alpha_s \in \pi_s(K^{n-1})$, $l=1, \cdots, n-1$, and let $i: \pi_{n-1}(K^{n-1}) \to \pi_{n-1}(K)$ be the injection. Using the formal pairs (α_l, α_{n-l}) for $1 \leq l \leq \frac{1}{2}n$, the author constructs a group U^{n-1} in which, roughly, a pair (α_l, α_{n-l}) represents the zero if $\alpha_l \circ \alpha_{n-l}$ is zero for algebraic reasons. A homomorphism w of U^{n-1} onto W_{n-1} is obtained. To investigate the kernel of iw, the author attaches products $S^l \times S^{n-l}$ as handles to a fixed S^n, obtaining "standard" manifolds; images of their basic n-cycles in K are called standard n-cycles. Main result: $iw(u)=0$ if and only if $w(u)$ is a homotopy boundary [H. Hopf, Comment. Math. Helv. 17, 307–326 (1945); MR 7, 36] of a standard n-cycle. This has as consequence: By suitable choice of basis $\{\xi_i\}$ in U^{n-1}, the elements of kernel iw can be represented as $\sum b_i\xi_i + \eta$, where (a) η belongs to the subgroup L generated by those pairs (α_l, α_{n-l}) in which at least one factor is homologically trivial in K and (b) if $\frac{1}{2}n$ is not an odd integer, the b_i are all Kronecker indexes of some standard n-cycle with cup products and with Pontrjagin squares in the cohomology of K. A condition under which kernel iw mod $L \neq 0$ is given. The last two results together yield a method for showing the existence of non-bounding (standard) n-cycles. As applications, results of Whitehead [Comment. Math. Helv. 22, 48–92 (1949); MR 10, 559] and Hirsch [C. R. Acad. Sci. Paris 228, 1920–1922 (1949); MR 11, 48] on π_3 of a simply connected complex are re-proved. In an appendix, the general method is generalized, and a theorem obtained which implies: Let $n>0$ be fixed; there is no finite complex Y with $\pi_i(Y)=0$ for all $i \neq 2n$, and with $\pi_{2n}(Y) \neq 0$ and infinite.

J. Dugundji.

Mycielski, J. Sur le coloriage des graphs. Colloq. Math. 3, 161–162 (1955).

The author gives a new construction for a finite graph which has no triangle and whose vertices cannot be coloured in n colours so that no two of the same colour are joined.

W. T. Tutte (Toronto, Ont.).

Rashevsky, N. Some theorems in topology and a possible biological implication. Bull. Math. Biophys. 17, 111–126 (1955).

In a previous paper [same Bull. 16, 317–348 (1954); MR 16, 386] the author defined a transformation of a directed linear graph P into a new graph $T(P)$: this was intended to represent the specialization of functions in biological tissues. Here it is shown that the fundamental group is unaltered by this transformation, but that another function named the 'point-base ratio' is increased.

C. A. B. Smith.

GEOMETRY

Aleksandrow, A. D. What is geometry? Wiadom. Mat. (2) 1, 4–46 (1955). (Polish)

Translation of the author's popular expository article in Bol'šaya Sovetskaya Ènciklopediya, 2d ed., vol. 10, pp. 533–549, Moscow, 1952.

Berghuys, J. J. W. L'objet de la géométrie. Synthese 9, 395–407 (1955).

✶Libois, P. Espaces, géométries, groupes. IIIᵉ Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 68–70. Fédération belge des Sociétés Scientifiques, Bruxelles.

✶Jacob, Maurice. La notion d'angle, la dimension "zéro" et les unités infinies. IIIᵉ Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 76–79. Fédération belge des Sociétés Scientifiques, Bruxelles.

Bilimović, Anton. Sur quelques propositions du sixième livre d'Eléments d'Euclide. Srpska Akad. Nauka. Zb. Rad. 43. Mat. Inst. 4, 67–71 (1955). (Serbo-Croatian. French summary)

Lorent, H. Une famille de triangles. Bull. Soc. Roy. Sci. Liège 24, 14–24 (1955).

Lorent, Henri. Sur les traces des pas d'Archimède. Mathesis 64, nos. 1–2, supplément, 1–7 (1955).

Lorent, H. Courbes associées à la conchoïde de Nicomède. Bull. Soc. Roy. Sci. Liège 24, 69–71 (1955).

Goormaghtigh, R. Sur le point de Miquel. Mathesis 64, 9–13 (1955).

de Majo, A. Faisceaux de sphères associés au tétraèdre. Mathesis 64, 13–19 (1955).

Saaty, T. L. The number of vertices of a polyhedron. Amer. Math. Monthly 62, 326–331 (1955).

If a simply-connected polyhedron has V vertices and F faces, we know that $V \leq 2F-4$ [E. Steinitz, Vorlesungen über die Theorie der Polyder, Springer, Berlin, 1934, p. 5]. Using Schläfli's generalization of Euler's formula, the author indicates how one can obtain such an inequality for an n-dimensional polytope having V vertices and F $(n-1)$-dimensional cells. After three misprints have been corrected, the result seems to be:

$$\frac{n-2}{2}V \leq \sum_{p=1}^{n/2} \binom{F}{n-2p} - F - 2^n + \binom{n+2}{2} \quad (n \text{ even}),$$

$$\frac{n-2}{2}V \leq \sum_{p=1}^{(n-1)/2} \binom{F}{n-2p} - 2^n + \binom{n+1}{2} \quad (n \text{ odd}).$$

The latter yields Steinitz's inequality when we set $n=3$. For greater odd values the author suggests, as an improvement,

$$\frac{n-2}{2}V \leq \sum_{p=1}^{(n-3)/2} \binom{F}{n-2p} - \frac{n-2}{2}F - 2^n + n(n+1).$$

H. S. M. Coxeter (Toronto, Ont.).

Fiedler, Miroslav. Geometry of the simplex in E_n. I. Časopis Pěst. Mat. **79**, 297–320 (1954). (Czech. Russian and English summaries)

The author investigates the existence and uniqueness of an n-dimensional simplex, given certain edges and certain angles (between faces). The main result is: The necessary and sufficient condition for the existence of a simplex whose edges O_iO_j have the prescribed lengths $\sqrt{e_{ij}}$ ($e_{ij}=e_{ji}$, $e_{ii}=0$) is $\sum_{i,j=1}^{n+1} e_{ij}x_ix_j<0$ whenever $\sum_{i=1}^{n+1} x_i=0$ (not all x_i being zero). A similar condition is obtained with prescribed angles; this implies that at least n angles must be acute; simplexes with exactly n acute and $n(n-1)/2$ right angles will be studied in later parts of this paper. *F. A. Behrend.*

Szász, Pál. Über die elementare Kreismessung. Mat. Lapok **5**, 73–78 (1954). (Hungarian. Russian and German summaries)

Let t_n denote the area of a regular n-sided polygon inscribed in a circle of unit radius and let T_n denote the area of a regular n-sided polygon circumscribed about this circle. Using the methods of elementary geometry the author obtains the following inequality for π:

$$\frac{3t_nT_n}{T_n+2t_n}<\pi<\frac{t_n+2T_n}{3}.$$

Combined with the inequality $1.732<\sqrt{3}<1.7321$, this gives $3.14<\pi<3.144$, and combined with the inequality $0.51763<\sqrt{(2-\sqrt{3})}<0.51764$ it gives $3.1413<\pi<3.1418$. *G. A. Dirac* (Vienna).

Locher-Ernst, L. Die zwölf Nabelpunkte des Ellipsoides. Elem. Math. **10**, 49–57 (1955).

In an earlier article the author has shown that Staudt's theory of imaginary elements in geometry becomes more readily approachable, if the directed elliptic involution, which in that theory represents an imaginary point, is symbolized by an arrow [Elem. Math. **4**, 97–105, 121–128 (1949); MR **11**, 385]. In the present paper some additions are made to this symbolic representation. The scheme is then made to parallel an analytic discussion of the known configuration consisting of the twelve umbilical points of an ellipsoid (eight of which are imaginary) and the eight imaginary lines each of which passes through three of the twelve points. Nine well drawn figures facilitate the reading of the discussion. The author concludes with the remark: "Unsere Ausführungen haben natürlich nur für denjenigen ein Wert, der ausser der Eleganz analytischer Entwicklungen auch die anschauliche Verarbeitung zu schätzen weiss". *N. A. Court* (Norman, Okla.).

Nyström, E. J. On special cones. Nordisk Mat. Tidskr. **3**, 27–32, 80 (1955). (Swedish. English summary)

Hohenberg, F. Herstellung von Perspektiven aux axonometrischen oder perspektiven Bildern. Elem. Math. **10**, 57–61 (1955).

Fempl, Stanimir. Über die Mantelflächen spezieller schiefer Kreiskegel. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **9**, 191–196 (1954). (Serbo-Croatian summary)

Niče, Vilko. Die Brennpunktsfläche der Kegelschnitte des Plückerschen Konoids. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **9**, 251–257 (1954). (Serbo-Croatian summary)

Niče, Vilim. Über gewisse zissoidale Begleitkurven und Begleitflächen aller Ordnungen. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **302**, 27–46 (1955). (Serbo-Croatian. German summary)

Niče, Vilim. Über das Gebüsch der durch 6 Punkte im Raum bestimmten Flächen 2. Ordnung. Rad Jugoslav. Akad. Znan. Umjet. Odjel Mat. Fiz. Tehn. Nauke **302**, 5–13 (1955). (Serbo-Croatian. German summary)

Hadamard, J. La géométrie non-euclidienne et les définitions axiomatiques. Acta Math. Acad. Sci. Hungar. **5**, supplementum, 95–104 (1954). (Russian summary)

This paper appeared first in a Hungarian translation [Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **3**, 199–208 (1953)] and was reviewed in MR **15**, 383.

Neumann, Maria. De l'interprétation de la géométrie de Lobatchewski sur un hyperboloïde. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 861–871 (1954). (Romanian. Russian and French summaries)

The author uses one sheet of the hyperboloid

$$-x^2-y^2+z^2=1$$

to represent the hyperbolic plane [cf. Coxeter, Amer. Math. Monthly **50**, 217–228 (1943), p. 223; MR **4**, 226]. Her results do not differ essentially from the classical formulae for distance and angle in terms of normalized canonical coordinates [see, e.g., Coxeter, Non-Euclidean geometry, Univ. of Toronto Press, 1942, pp. 158, 210, 249, 255; MR **4**, 50]. *H. S. M. Coxeter* (Toronto, Ont.).

Szász, Paul. Elementargeometrische Herstellung des Klein-Hilbertschen Kugelmodells des hyperbolischen Raumes. Acta Sci. Math. Szeged **16**, 1–8 (1955).

The author proves the consistency of three-dimensional hyperbolic geometry in an elementary manner. He uses the Klein-Hilbert spherical model and avoids the use of the concept of collineation. *E. Lukacs* (Washington, D. C.).

Szász, Paul. Diverses présentations élémentaires de la trigonométrie hyperbolique. Acta Math. Acad. Sci. Hungar. **5**, supplementum, 105–116 (1954). (Russian summary)

This is the French version of an earlier, expository lecture of the author [Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **3**, 209–218 (1953); MR **15**, 461]. *E. Lukacs.*

Szász, Paul. Elementargeometrischer Beweis der Widerspruchsfreiheit der hyperbolischen Raumgeometrie mit Hilfe des Poincaréschen Halbraumes. Acta Math. Acad. Sci. Hungar. **5**, 255–261 (1954). (Russian summary)

When the planes of hyperbolic space are represented by hemispheres and half-planes orthogonal to a fixed plane in Euclidean space, with congruence defined in terms of cross-ratio, most of Hilbert's axioms for hyperbolic space are obviously satisfied. The author uses stereographic projection to verify the only difficult one (in which a triangle is determined by two sides and the included angle). *H. S. M. Coxeter* (Toronto, Ont.).

D'Antona, Giuseppina. Considerazioni varie sulla metrica angolare iperbolica. Matematiche, Catania **9**, 7–22 (1954).

Representing the absolute conic of the hyperbolic plane by an ellipse with semiaxes a and b in the Euclidean plane,

the author compares the Euclidean and hyperbolic metrics. At a point within the ellipse, he considers the Euclidean angle h_e formed by a line parallel to the minor axis and an oblique line, comparing h_e with the hyperbolic angle h_i formed by the same two lines. He shows that the locus of points where both h_e and h_i have assigned values in a quartic (in the Euclidean plane) passing through the ends of the major axis and touching the ellipse at another pair of diametrically opposite points. It has one or two circuits according as $a \tan h_i$ is $\leqq$ or $>b \tan h_e$, with a node in the case of equality. In particular, when the same value is assigned to both h_i and h_e, there are two circuits, one through each focus. *H. S. M. Coxeter* (Toronto, Ont.).

Edge, W. L. Line geometry in three dimensions over GF(3), **and the allied geometry of quadrics in four and five dimensions.** Proc. Roy. Soc. London. Ser. A. **228**, 129–146 (1955).

In this sequel to an earlier paper [same Proc. **222**, 262–286 (1954); MR **15**, 818] the author describes the line geometry of a projective three space S over a field K of three marks $0, 1, -1$, and the related geometry of quadrics in [5] or [4] over K. Pluecker coordinates of the 130 lines of S define the 130 points L on a quadric Ω in projective 5-space defined by $j=0$, where $j=p_{14}p_{23}+p_{24}p_{31}+p_{34}p_{12}$. The 234 points A in [5] not on Ω are divided into equal "positive" and "negative" sets of 117 points according as $j=1$ or -1, each set invariant under $PO_3(6, 3)$. Each A is the pole of a prime α in [5] that intersects Ω in a quadric ω whose 40 points m correspond to the 40 lines of a screw (non-degenerate linear complex) σ in S. The screw σ and each of its 4 line reguli are given a sign determined by A. Each hyperboloid in S has two complementary reguli of opposite sign. Forty five of the points of α not on the quadric ω lie on chords to Ω through A. Each of these points is a vertex of three different pentagons $\wp$ in α that are self polar with respect to ω. Each of the 27 pentagons so formed has 10 edges skew to ω, and 10 planes each containing 4 points of ω that correspond to a regulus in S. The configuration of 27 self-polar pentagons with their 45 vertices is shown to be isomorphic with the configuration of the 27 lines and 45 triangles on a cubic surface, and with the configuration built by Baker on the 45 nodes of the Burkhardt primal. Thus in a natural way a subgroup $\omega(5, 3)$ of index 117 in $PO_3(6, 3)$ is identified with the cubic surface group of order 51840, and the four classes of involutions in this group are identified with the four types of harmonic inversion in α whose fundamental spaces are polars in regard to ω. *J. S. Frame.*

Convex Domains, Extremal Problems, Integral Geometry

Süss, Wilhelm. Ueber Parallelogramme und Rechtecke, die sich ebenen Eibereichen einbeschreiben lassen. Rend. Mat. e Appl. (5) **14**, 338–341 (1955).

A proof of the existence of an inscribed rectangle of area $A/2$, in a given plane convex figure of area A, is derived from that of an inscribed parallelogram of area $A/2$ with a horizontal pair of sides. The author also corrects a corresponding statement of Radziszewski in 3-space [C. R. Acad. Sci. Paris **235**, 771–773 (1952); cf., however, the latter's further paper, Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **6**, 5–18 (1954); MR **14**, 896; **15**, 981 (the second review contains some obvious misprints: A for V and $aA/9$ for $2V/9$)]. *L. C. Young* (Madison, Wis.).

Ohmann, D. Eine Verallgemeinerung der Steinerschen Formel. Math. Ann. **129**, 209–212 (1955).

In pursuance of his program of generalization [Math. Ann. **127**, 1–7 (1954); MR **15**, 738], the author shows that Steiner's formula, for convex Euclidean sets, is substantially the case of equality in a general inequality for compact Euclidean sets. The presentation suffers a little from the author's use of a suffix which is sometimes a positive real and sometimes a unit vector. *L. C. Young.*

Berge, Claude. Sur un théorème de la convexité régulière non linéaire. C. R. Acad. Sci. Paris. **238**, 2485–2486 (1954).

Theorem IV of the author's previous paper [Bull. Soc. Math. France **82**, 301–315 (1954); MR **16**, 717 (the theorem quoted in the first paragraph of the review)] is sharpened via a strengthened hypothesis, viz., lower semicontinuity of the functionals f is replaced by continuity. The separation of the convex sets which is then obtained is strict: there is an $f \varepsilon F$ and an α such that $f(x)>\alpha$, $x \varepsilon C$; $f(x)<\alpha$, $x \varepsilon C'$. *B. Gelbaum* (Minneapolis, Minn.).

Sz.-Nagy, Gyula. Konvexes Polyeder als geometrischer Ort. Acta Math. Acad. Sci. Hungar. **5**, 165–167 (1954). (Russian summary)

The following theorem proved in this paper. Let $E_1, \cdots, E_n$ be n fixed planes, not all parallel to a given line, in Euclidean space. The geometric locus $G(k)$ of the point P, for which $A(P)$, the sum of the distances of P from $E_1, \cdots, E_n$, takes the constant value $k>k_0=\min A(P)$, is the boundary of a convex polyhedron whose vertices lie on the planes $E_1, \cdots, E_n$. The "kernel" $G(k_0)$ of the locus $G(k)$ is a convex polyhedron which may also degenerate to a polygon, to a straight line, or to a point. *T. L. Saaty.*

Moroń, Z. On the dissection of rectangles into squares. Wiadom. Mat. (2) **1**, 75–94 (1955). (Polish)

Expository article consisting mainly of survey of published material on this topic. *L. C. Young.*

Schütte, Kurt. Überdeckungen der Kugel mit höchstens acht Kreisen. Math. Ann. **129**, 181–186 (1955).

It was shown by Fejes Tóth [Lagerungen in der Ebene, auf der Kugel und im Raum, Springer, Berlin, 1953, pp. 114, 147; MR **15**, 248] that the minimum radius r for N equal small circles covering the unit sphere satisfies the inequality

$$r \geqq \arc\cos(3^{-1/2}\cot\omega), \quad \omega=N\pi/6(N-2),$$

this lower bound being attained when $N=3$, 4, 6 or 12. The author proves that $r=\arctan 2$ when $N=5$, and $\arctan(\sqrt{5}-1)$ when $N=7$. In either case (as also when $N=6$) the centers of the small circles are the vertices of a dipyramid. He also finds a remarkably symmetrical arrangement of 8 circles of radius 48° 9′. (Thus, for $N=8$, $46° 31' \leqq r \leqq 48° 9'$, with a strong expectation that this upper bound is the true value.) The centers of the eight circles are the vertices of a polyhedron whose twelve isosceles triangular faces are arranged like the twelve equilateral triangular faces of the "Siamese dodecahedron" of Freudenthal and van der Waerden [Simon Stevin **25**, 115–121 (1947); MR **9**, 99]. *H. S. M. Coxeter* (Toronto, Ont.).

Leja, F. Span and extremal points of a set. Prace Mat. 1, 56–70 (1955). (Polish. Russian and English summaries)

Survey of published material connected with a formal generalization, given by the author in 1933, of the notion of transfinite diameter. The survey concludes with five problems to be solved. *L. C. Young* (Madison, Wis.).

Kárteszi, Franz. Extremalaufgaben über endliche Punktsysteme. Publ. Math. Debrecen 4, 16–27 (1955).

The author considers, in the real affine plane, a system of n points, no three collinear. A point P, not on any of the $\binom{n}{2}$ joining lines, is said to be covered k times if it lies within just k of the $\binom{n}{3}$ triangles. For a given system and various positions of P, k has a certain maximum. For various systems of n points, this maximum k has a maximum $W(n)$ and a minimum $V(n)$; thus $W(3)=V(3)=1$, $W(4)=V(4)=2$, $W(5)=V(5)=5$, $W(6)=V(6)=8$. By an ingenious argument, the author proves that $W(n)=n(n^2-1)/24$ or $n(n^2-4)/24$ according as n is odd or even; e.g. $W(7)=14$. He exhibits a system of 7 points such that there is a region where $k=13$ but no place where $k=14$, thus showing that $V(7)<W(7)$. *H. S. M. Coxeter* (Toronto, Ont.).

Karush, W., and Wolfsohn, N. Z. The distance to the origin of a certain point set in E^n. Proc. Amer. Math. Soc. 6, 323–332 (1955).

It is proved that the minimum value of $\sum_{i=1}^{m+1} a_i^2$ for real $a_1, \cdots, a_{m+1}$, subject to the conditions

$$\sum_{i=1}^{m+1} i^k a_i = 1 \quad (k=0, 1, \cdots, q;\ 0 \leqq q < m),$$

is $1-[m(m-1)\cdots(m-q)]/[(m+1)(m+2)\cdots(m+q+1)]$. The problem has a statistical origin. The proof depends on the evaluation of the determinant $|S_{k+j}(m)|_{k,j=0}^{q}$, where $S_k(m)=\sum_{i=1}^{m+1} i^k$. *F. F. Bonsall* (Newcastle-on-Tyne).

Leichtweiss, Kurt. Zwei Extremalprobleme der Minkowski-Geometrie. Math. Z. 62, 37–49 (1955).

Let K be any (bounded and closed) convex body with diameter Δ of an n-dimensional Minkowski space M_n, and define $\rho_K=\min_{x_0\in M_n}(\sup_{x\in K} x_0x)$, where x_0x denotes the distance of x_0, x. The author proves by geometric arguments four theorems. Theorem 1 shows that $\rho_K \leqq (n/n+1)\Delta$, and the factor $n/n+1$ is the best possible. This extension to Minkowski space of Jung's theorem [J. Reine Angew. Math. 123, 241–257 (1901)] was first proved by Bohnenblust [Ann. of Math. (2) 39, 301–308 (1938), pp. 305–307] in a different manner and with the assumption that the indicatrix Σ is centrally symmetric (not assumed in the present paper). Theorem 2 characterizes an n-simplex S of M_n by the property $\rho_S=(n/n+1)\Delta$. Theorems 3 and 4 concern the largest homothetic image of the indicatrix Σ of M_n that is contained in K. It is shown that the radius ρ of such an image is related to the suitably defined Minkowski breadth b of K by the inequality $\rho \geqq b/(n+1)$, and in case equality holds then K is an n-simplex. *L. M. Blumenthal* (Columbia, Mo.).

Pleijel, Arne. On convex curves. Nordisk Mat. Tidskr. 3, 57–63, 80 (1955). (Swedish. English summary)

Let A be the area of a plane convex figure whose contour has length L and bounds r, R for its radius of curvature. The author obtains the best value $\pi(4-\pi)$ of the constant K in the inequality $L^2-4\pi A \leqq K(R-r)^2$ due to Bottema [Akad. Wetensch. Amsterdam. Proc. 36, 442–446 (1933)]. He also gives, in terms of L and R, an estimate for A and bounds for the distance of parallel tangents. *L. C. Young* (Madison, Wis.).

Algebraic Geometry

Balsimelli, Pio. Una trasformazione quadratica biduale e la relativa trasformazione cremoniana nell'S_3 ambiente della congruenza Γ. Rend. Accad. Sci. Fis. Mat. Napoli (4) 21, 25–30 (1954).

***Lorent, H. Sur les éléments singuliers des lignes et des surfaces.** III[e] Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 91–92. Fédération belge des Sociétés Scientifiques, Bruxelles.

Godeaux, Lucien. Sulla struttura di un punto di diramazione di una superficie algebrica multipla di ordine 31. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) 18(87), 619–626 (1954).

***Godeaux, Lucien. Surfaces algébriques s'osculant le long de courbes.** III[e] Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 97–99. Fédération belge des Sociétés Scientifiques, Bruxelles.

Franchetta, A. Sulle serie lineari razionalmente determinate sulla curva a moduli generali di dato genere. Matematiche, Catania 9, 126–147 (1954).

The author proves the following theorems. I. A complete linear series determined rationally on a general canonical curve of genus $p>2$ is either the canonical series or some multiple of the canonical series. II. A complete linear series determined rationally on the general curve of a non-special family of curves of genus $p>1$ is a linear combination of the canonical series and the series of hyperplane sections. *J. A. Todd* (Cambridge, England).

Martinelli, Enzo. Sulle intersezioni delle curve analitiche complesse. Rend. Mat. e Appl. (5) 14, 422–430 (1955).

The author proves the following theorem: Let S be a simple closed hypersurface in R_4, and let D denote the domain inside S. Let F and G be two analytic surfaces in $S\cup D$ without common points in D. Each of the surfaces F and G intersects S in a system of closed curves, and the linking number of these two systems will be equal to the number of intersections between F and G in D, the intersections being counted with their multiplicities. This result leads to a convenient definition of the multiplicity of a point of intersection between two analytic surfaces. On the other hand, it also leads to a simple proof of Bézout's theorem. *H. Tornehave* (Copenhagen).

Morelock, J. C., and Perry, N. C. On algebraic surfaces termwise invariant under cyclic collineations. Canad. J. Math. 7, 204–207 (1955).

On considère ici une surface algébrique F, de l'espace ordinaire, qui soit transformée en elle-même par l'homographie T: $(x_1', x_2', x_3', x_4')=(x_1, Ex_2, E^2x_3, E^3x_4)$, où $E^p=1$ et p est un nombre premier, dans l'hypothèse que chacun des termes $x_1^a x_2^b x_3^c x_4^d$ de l'équation de F soit invariant pour T. Si l'ordre de F est un multiple mp de p, cela revient à dire que: $a+b+c+d=mp$, $b+2c+3d=kp$. On en déduit très simplement les conditions suivantes: $b\equiv d-2a$, $c\equiv a-2d$

(mod p). En particulier, si $m=1$, on trouve encore des conditions pour a, d; on en conclut que, dans ce cas, et si $p>3$, le nombre des termes dans l'équation de F est $\frac{1}{6}(p^2+6p+17)$.
E. Togliatti (Gênes).

Northcott, D. G. Analytically biregular mappings. Proc. London Math. Soc. (3) **5**, 219–237 (1955).

Suppose that U is an algebraic variety over an arbitrary algebraically closed field with the subvarieties V and W for whose intersection T is a proper component. Assume that $P \in T$ is a simple point on U. The author establishes sufficient conditions on a rational transformation φ of U on $U'=\varphi(U)$ such that the intersection multiplicity of V and W along T on U is retained for the corresponding transforms by φ. It is assumed that the image $P'=\varphi(P)$ is a simple point on U'. If $\mathfrak{o}$ ($\mathfrak{o}'$) denotes the local ring of U (U') at P (P') with the maximal ideal $\mathfrak{m}$ ($\mathfrak{m}'$), then $\mathfrak{o}'\subseteq\mathfrak{o}$ and $\mathfrak{m}\cap\mathfrak{o}'=\mathfrak{m}'$. The mapping φ is called analytically biregular if (i) $\mathfrak{o}\mathfrak{m}'$ is $\mathfrak{m}$-primary, and (ii) the quotient fields of the $\mathfrak{m}$ ($\mathfrak{m}'$)-adic completions of $\mathfrak{o}$ and $\mathfrak{o}'$ coincide. Then φ is shown to preserve the above mentioned intersection multiplicity if it is biregular at the point P through which T passes and if P is the only point on V which is transformed into P' by φ. The proof of this theorem uses a description of analytic biregularity in terms of the rank of a matrix involving partial derivatives of the equations describing φ and U. The main tool employed by the author is Weil's theory of specializations and a reduction of the theorem to Weil's theorem on biregular transformations [Foundations of algebraic geometry, Amer. Math. Soc. Colloq. Publ., v. 29, New York, 1946; Th. 10, p. 159; MR **9**, 303]. *O. F. G. Schilling.*

Blanchard, André. Un théorème sur les automorphismes d'une variété algébrique projective. Application aux fibrés et à une conjecture d'Igusa. C. R. Acad. Sci. Paris **240**, 2198–2201 (1955).

Let V be a complex algebraic variety, $A(V)$ its Albanese variety, and G^0 the connected group of all automorphisms of V. Let Φ be a finite subgroup of the group of translations of $A(V)$ and N_Φ the inverse image of Φ in G^0. It is proved that there exists an imbedding of V in the complex projective space such that the automorphisms of N_Φ are induced by homographic transformations of the space. Using this, the author derives the following theorem: Let $E(B, F)$ be a compact complex analytic fiber space with base space B and fiber F, and with a connected structural group. In order that E be an algebraic variety it is necessary and sufficient that the following conditions be satisfied: 1) B and F are algebraic varieties; 2) $b_1(E)=b_1(B)+b_1(F)$, where b_1 denotes the first Betti number of a space; 3) The Albanese variety $A(V)$ is an abelian variety.

The author gives an example of a Kähler variety, which cannot be transformed by monoidal transformations into a bundle of complex tori over an algebraic variety, thus disproving a conjecture of Igusa. *S. Chern.*

Gauthier, Luc. Nombres de Betti des intersections complètes de formes quadratiques. C. R. Acad. Sci. Paris **240**, 1851–1853 (1955).

Let $V=V_{n-p}^{2^p}$ be the complete intersection of p quadratic forms in a complex projective space of n dimensions. The author determines its Betti numbers by using a theorem of Lefschetz. The results are found in agreement with those obtained by Godeaux [Acad. Roy. Belg. Bull. Cl. Sci. (5) **30**, 262–269 (1945); MR **8**, 89]. They also confirm a conjecture of Weil on the ζ-function associated to the variety V^4_{n-2} over a finite field. *S. Chern* (Chicago, Ill.).

Chevalley, C. Sur la théorie des variétés algébriques. Nagoya Math. J. **8**, 1–43 (1955).

L'auteur donne une définition des variétés algébriques qui est essentiellement équivalente à la définition des "Variétés Abstraites" de A. Weil, et qui possède sur cette dernière les deux avantages suivants: 1) la décomposition de la variété en morceaux affines n'est pas donnée à l'avance; 2) une variété est un ensemble de points, ce qui permet d'utiliser librement le langage de la théorie des ensembles. La définition en question s'appuie sur le fait que l'ensemble $M(V)$ des anneaux locaux absolus (c.à.d. construits sur le domaine universel K) de tous les points de V est un invariant birégulier de V qui détermine V à un isomorphisme birégulier près. Etant donné un corps de fonctions algébriques L sur K, on appelle modèle affine de L l'ensemble des anneaux de fractions, relatifs à ses idéaux maximaux, d'un anneau de la forme $K[x_1, \cdots, x_n]$ admettant L pour corps des fractions; un modèle de L est alors un ensemble M de sous anneaux locaux de L, tel que M soit réunion finie de modèles affines, et que, étant donnés deux éléments distincts A et A' de M, l'idéal de A [A'] engendré par les idéaux maximaux de A et de A' contienne toujours 1. Ceci étant une variété V est un ensemble V de points muni d'une famille L d'applications de parties de V dans K tels que: 1) L soit un corps de fonctions algébriques sur K; 2) si, pour $P \in V$, on désigne par $A(P)$ l'ensemble des éléments de L qui sont définis en P, $A(P)$ est un anneau local et l'ensemble des $A(P)$ est un modèle du corps L.

L'auteur définit alors la topologie de Zariski de V comme étant celle engendrée par les domaines de définition des éléments de L; il montre que toute famille non vide d'ouverts y admet un élément maximal, et déduit de là la décomposition de toute partie de V en parties relativement fermées et irréductibles. Il caractérise les parties de V qui admettent une structure induite de variété algébrique: ce sont celles qui sont irréductibles et ouvertes dans leur adhérence (ainsi un espace affine est une sous variété de l'espace projectif correspondant, ce qu'il n'était pas dans la théorie de Weil où toutes les sous variétés étaient des fermés). Etude des produits et des correspondances, et liens de celles ci avec les extensions composées de corps.

On appelle partie constructible d'une variété V toute réunion finie de sous variétés de V, ou encore réunion finie d'intersections d'un fermé et d'un ouvert. La famille des parties constructibles de V est stable pour les opérations ensemblistes fondamentales. De plus, si C est une correspondance entre deux variétés U et V et si U' est une partie constructible de U, alors l'ensemble V' des points de V qui correspondent par C à au moins un point de U' est un ensemble constructible de V. Ceci permet d'utiliser commodément le langage de la théorie des ensembles, en dépit du fait que la projection (ensembliste) d'une variété ne soit pas nécessairement une variété.

Enfin, étant donné un sous corps k du domaine universel K, l'auteur définit la notion de k-variété, et montre que pour toute variété V, il existe une extension k de type fini du corps premier telle que la structure de variété de V soit sousjacente à une structure de k-variété de V. On peut ainsi parler de corps de définitions et de points génériques, et retrouver toute la commodité technique qu'apportent ces notions. *P. Samuel* (Clermont-Ferrand).

Differential Geometry

✶Vrânceanu, Gheorghe. Lecţii de geometrie diferenţială. Vol. I. Congruenţe. Forme ale lui Pfaff. Grupuri continue. Invarianţi şi echivalenţă. Spaţii cu conexiune afină. Spaţiile lui Riemann. Spaţii cu conexiune proiectivă. [Lectures on differential geometry. Vol. I. Congruences. Pfaffian forms. Continuous groups. Invariants and equivalents. Affinely connected spaces. Riemannian spaces. Projectively connected spaces.] Editura Academiei Republicii Populare Române, 1952. 344 pp. 7.50 Lei.

This first volume is a translation into Rumanian of "Leçons de géométrie différentielle" [Bucarest, 1947; MR **9**, 532], with some minor changes. *D. J. Struik.*

✶Vrânceanu, Gheorghe. Lecţii de geometrie diferenţială. Vol. II. Spaţiile lui Cagan. Conexiuni conforme. Tensori de al doilea ordin. Subspaţii. Spaţii neolonome. Ecuaţii cu derivate parţiale de al doilea ordin. Geometrie diferenţială globală. [Lectures on differential geometry. Vol. II. Kagan spaces. Conformal connections. Tensors of second order. Subspaces. Nonholonomic spaces. Partial differential equations of the second order. Global differential geometry.] Editura Academiei Republicii Populare Române, 1951. 398 pp. 150 Lei.

This second volume contains material which does not appear in the French version of the book, which is only the translation of the first Romanian volume. There are six chapters. The first deals with the subprojective spaces of Kagan, with special attention paid to automorphisms, and especially to the work of I. P. Egorov [Dokl. Akad. Nauk SSSR (N.S.) **57**, 867–870 (1947); **73**, 265–267 (1950); MR **9**, 468; **12**, 636] and the author himself [Acad. Repub. Pop. Române. Stud. Cerc. Mat. **2**, 387–444 (1951); MR **16**, 623]. This chapter should be studied together with the theory of subprojective spaces presented in the second edition of Schouten's "Ricci-calculus" [Springer, Berlin, 1954, MR **16**, 521]. The second chapter contains a study of spaces of conformal connection, where we find again studies on transformations by which such spaces pass into themselves.

Chapter III is on "tensors of order two", and has an investigation of spaces Q_n in which an arbitrary bivalent tensor $A_{ij}=a_{ij}+b_{ij}$ is given, where $a_{ij}=a_{ji}$, $b_{ij}=-b_{ji}$, and hence two invariant forms $\phi=a_{ij}dx^idx^j$, $\psi=b_{ij}\delta x^i\delta x^j$ exist. Simplectic spaces have only the alternate form. Taken up are results by the author [C. R. Acad. Sci. Paris **229**, 336–338 (1949); MR **11**, 134], Ehresmann and Libermann [ibid. **227**, 420–421 (1948); MR **10**, 122] and C. T. Yen [ibid. **227**, 817–819, 1203–1204 (1948); MR **10**, 480]. Here we find discussions on spaces with a given contravariant tensor A^{ij}, not necessarily symmetrical, and of spaces with mixed tensor a_j^i. In this last case the space admits systems of congruences defined by $(a_j^i-\rho\delta_j^i)dx^j=0$, and the investigations lead into fields covered by the author's work on non-holonomic varieties [published, e.g., in Mémor. Sci. Math. no. **76** (1936)].

Chapter IV deals with subspaces of X_n of different types, e.g. with tensors a_j^i or A^{ij}, or with affine connection. Here we find discussions on the rigidity conditions of an affine A_m in a euclidean affine E_n. Then follow non-holonomic spaces, among them the X_n^{n-2}, and X_n^m in A_n, projectivities of Bompiani [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **28**, 283–291, 292–301 (1938)] and some remarks on X_n^m in a projective P_n and a riemannian V_n.

Chapter V deals with a partial differential equation of order two $F(x, y, z, p, q, r, s, t)=0$ with the three Pfaffians

$$ds^1=dz-pdx-qdy=0,\quad ds^2=dp-rdx-sdy=0,$$
$$ds^3=dq-sdk-tdy=0,$$

which determine a non-holonomic X_7^4 in the

$$X_7(x, y, z, p, q, u, v),$$

r, s, t being replaced by the auxiliary variables u, v. This allows the author to bring together a multitude of known and new results, among them the theorem of Cartan on systems of Pfaff in 5 variables [Ann. Sci. Ecole Norm. Sup. (3) **27**, 109–192 (1910)], work on Bäcklund transformations, the theorem of Ţiţeica on Laplace equations with constant absolute invariants [C. R. Acad. Sci. Paris **200**, 191–192 (1935)], and research by the author himself [e.g. Mathematica, Timişoara **20**, 98–112 (1944); MR **6**, 229].

The last chapter contains several sections on differential geometry in the large such as topological manifolds with general affine connection and with constant affine connection. This leads up to such theorems as that of Mocanu [Com. Acad. R. P. Române **1**, 239–243 (1951)]. Topological manifolds with affine connection and self-parallel curves are next considered, with an application to subprojective spaces. The book ends with the Gauss-Bonnet theorem. We hope that this book, with its many new or relatively unknown results, will soon be available in translation.

D. J. Struik (Cambridge, Mass.).

Vincze, Stefan. Bemerkungen zur Differentialgeometrie der Raumkurven. Publ. Math. Debrecen **4**, 61–69 (1955).

Let L be a space curve with a continuous second derivative with respect to its arc length s at point P. Let $\mu(s)$ be a mass distribution on L, μ having a continuous second derivative. Let Q_1 and Q_2 be two points on L corresponding to $s=s_1$ and s_2. Let S be the center of mass of the arc Q_1Q_2. Let S' be the center of mass of the system consisting of Q_1 and Q_2 with weights $\mu(s_1)$ and $\mu(s_2)$. Then

$$\lim_{Q_1, Q_2\to P}\frac{\overrightarrow{SS'}}{\widehat{Q_1Q_2}^2}=\frac{1}{12}\left(\mathbf{k}+2\frac{\mu'}{\mu}\mathbf{t}\right),$$

where $\mathbf{t}$ is the unit tangent vector of L at P and $\mathbf{k}=d\mathbf{t}/ds$ is the curvature vector at P.

Let F be the surface consisting of the centers of mass S of the various arcs $\widehat{Q_1Q_2}$. The author finds a differential equation satisfied by F, shows that L is an asymptotic line on F, and proves the following theorem: Let L have a continuous fourth derivative at P, and let the mass distribution be homogeneous. Let $K(Q_1, Q_2)$ and $H(Q_1, Q_2)$ be the Gaussian and mean curvatures of F at the point $S(Q_1, Q_2)$. Let $1/\rho$ and $1/\tau$ be the curvature and torsion of L. Then

$$\lim_{Q_1, Q_2\to P}K(Q_1, Q_2)=-\frac{1}{\tau^2},\quad \lim_{Q_1, Q_2\to P}H(Q_1, Q_2)=-\tfrac{3}{8}\rho^2\frac{d}{ds}\left(\frac{1}{\rho^2\tau}\right).$$

Finally, let $f(\mathbf{r})$ be a scalar function of the position vector $\mathbf{r}$, f having continuous first partial derivatives. Consider a sphere of radius ρ with center at P. Let $\exp f(\mathbf{r}(Q))$, Q a point of the sphere, define a mass distribution. Let S be the center of mass of the sphere. Then $\lim_{\rho\to 0}\overrightarrow{PS}/\rho^2=\tfrac{1}{3}\operatorname{grad} f(\mathbf{r})$.

For the most part, the proofs of these theorems consist of straightforward computations with Taylor series expansions.

A. Schwartz (New York, N. Y.).

Scherrer, W. Die Grundgleichungen der Flächentheorie. I. Comment. Math. Helv. 29, 180–198 (1955).

From the author's Introduction: Der Zweck der vorliegenden Abhandlung ist es nun, zu zeigen, dass es auch vorteilhaft ist, die ganze Darstellung einer Fläche auf ein normiertes Dreibein mit einem schiefen Winkel zu gründen. Man erhält dann eine jedem Parameternetz anpassbare Darstellung, bei welcher jede auftretende Grundgrösse eine unmittelbare geometrische Bedeutung besitzt.

V. Hlavatý (Bloomington, Ind.).

Mayer, O. Application of the tensor calculus to the theory of surfaces in three-dimensional Euclidean space. Gaz. Mat. Fiz. Ser. A. 7, 101–114 (1955). (Romanian)

Expository paper.

Rembs, Eduard. Randvorgaben bei infinitesimaler Verbiegung konvexer Flächen. Arch. Math. 6, 55–58 (1954).

This paper is concerned with the effect of the infinitesimal bending of a convex surface in three-dimensional space upon a plane curve C on the surface. Let s represent the arc corresponding to a variable point P of C and $(x_1(s), x_2(s))$ the coordinates of P in any Cartesian frame which has been chosen for the plane of C. Then, if δk is the variation in the curvature of C which results from the infinitesimal bending, the three following relationships are valid:

$$\oint \delta k \, ds=0, \quad \oint \delta k \, x_1(s)ds=0, \quad \oint \delta k \, x_2(s)ds=0.$$

From these formulas it is deduced that the curvature of C must be stationary at least four points. *A. Douglis.*

Brauner, Heinrich. Quadriken als Bewegflächen. Monatsh. Math. 59, 45–63 (1955).

Die ∞^3 Kegelschnitte, die auf einer Quadrik liegen, sind (da die Gestalt eines Kegels durch zwei Zahlen bestimmt ist) zu je ∞^1 untereinander kongruent. Es muss also möglich sein einen Kegelschnitt auf der Fläche zu bewegen, d.h. es gibt Erzeugungen einer Quadrik als Bewegfläche. Verf. geht auf die Bewegungen selbst nicht weiter ein, gibt aber folgende Erweiterung an. Einer Quadrik kann man längs jedem ihrer Kegelschnitte ∞^1 Flächen zweiter Ordnung anschreiben; unter diesen ∞^4 sind je ∞^1 kongruente, d.h. die Quadrik kann (auf ∞^3 Arten) als Hüllgebilde einer starren Fläche 2. Ordnung erzeugt werden. Verf. gibt eine analytische Behandlung dieser Fragen und bestimmt nicht nur die kongruenten sondern auch die ähnlichen und die volumsgleichen angeschriebenen Quadriken. *O. Bottema* (Delft).

Backes, F. Sur un cas de correspondance avec orthogonalité des éléments. Acad. Roy. Belg. Bull. Cl. Sci. (5) 41, 101–105 (1955).

By extending Schwarz's formula which gives the adjoint surface of a given minimal surface, the author associates to every surface Σ with constant mean curvature H $(H\neq 0)$, another surface Σ_1 and shows that Σ and Σ_1 possess some of the properties known to exist between a minimal surface and its adjoint, namely: a) Corresponding directions at corresponding points of the two surfaces are orthogonal. b) To the asymptotic lines of Σ_1 correspond the lines of curvature of Σ. c) The developable surfaces of the rectilinear congruence defined by drawing the parallel through every point P of Σ to the normal at P_1 to Σ_1, P and P_1 being corresponding points, cut Σ along its lines of curvature, and Σ is the middle surface of this congruence. Finally, the particular solution of Moutard's equation, corresponding to the case mentioned above, is determined. *F. Şemin.*

Kovancov, N. I. Two propositions about limit points and foci of nonholonomic congruences. Uspehi Mat. Nauk (N.S.) 10, no. 1(63), 113–116 (1955). (Russian)

An equation $P\,dx+Q\,dy+R\,dz=0$, P, Q, R functions of x, y, z, the Cartesian coordinates of a euclidean space, determines at each point a plane element, the normals to which are the lines of a linear complex. The lines in the neighborhood of a given line of this complex can be considered, in so far as properties of the first order are concerned, as lines of a linear complex. Some theorems on limit points and foci of the classical theory of linear complexes are generalized to this case. *D. J. Struik* (Cambridge, Mass.).

Kovancov, N. I. Pairs of complexes of a projective rotation. Dokl. Akad. Nauk SSSR (N.S.) 100, 863–866 (1955). (Russian)

The equations of the principal surfaces of a ruled complex are almost identical with the characteristic equation of quadric surfaces. A general complex has a characteristic equation with three different roots so that there exist three principal surfaces. The case of three equal roots corresponds to a linear complex so that all its ruled surfaces are principal. This paper is primarily concerned with the case in which the roots $S_1=S_2\neq S_3$ are the characteristic roots. It is this complex that the author calls a complex of projective rotation and shows that each principal surface, in this case, belongs to a pencil of linear complexes. *M. S. Knebelman.*

Tuganov, N. G. On a congruence of lines of the second order in a three-dimensional projective space. Dokl. Akad. Nauk SSSR (N.S.) 100, 13–15 (1955). (Russian)

A congruence C^2 in a three-dimensional projective space being given, the repère mobile may be so normalized that the lines of the congruence are given by $x_1x_2+x_2x_3+x_3x_1=0$. With this normalization the coefficients of the displacement ω_j^i, $i, j=1, 2, 3$, are subject to a number of relations whose solution depends on six arbitrary functions of two arguments. The geometrical interpretation of this fact is that, six arbitrary surfaces being given, they determine a line congruence C^2. By means of a representation of C^2 in a projective space of five dimensions the author constructs a linear congruence tangent to C^2. *M. S. Knebelman.*

Vincensini, Paul. Su una famiglia di reti geodetico-planari. Boll. Un. Mat. Ital. (3) 10, 11–16 (1955).

The author continues his study of nets on surfaces, of which one family of ∞^1 curves is plane, and the other family consists of geodesics. These nets were already introduced [C. R. Acad. Sci. Paris 239, 1113–1114 (1954); MR 16, 400]. He shows the use of the correspondence between lines in E_3 and points in euclidean E_4 developed earlier [ibid. 240, 481–483 (1955); MR 16, 622] and points to the relationship, by means of Lie transformations, between these nets and certain surfaces discussed by L. Godeaux [Univ. e Politec. Torino. Rend. Sem. Mat. 13, 39–46 (1954); MR 16, 885]. *D. J. Struik* (Cambridge, Mass.).

Petrescu, Şt. La classification des espaces P_2 à connexion projective. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 5, 485–491 (1953). (Romanian. Russian and French summaries)

The investigations of the author in Acad. Repub. Pop. Române. Stud. Cerc. Mat. 3, 529–558 (1952) [MR 16, 624]

are continued for P_2 with both non-zero curvature and torsion. The P_2 admits congruences (λ^a), $\alpha=1, 2, 0$, given by the Pfaffians

$$ds^a=\lambda_i^a(x^1, x^2)dx^i;\quad ds^0=dx^0+\lambda_i^0(x^1, x^2)dx^i;\qquad i, j=1, 2;\ a, b, c=1, 2,$$

with transformation group

$$d\bar{s}^a=c_b^a(x^1, x^2)ds^b;\quad d\bar{s}^0=ds^0+c_b^0(x^1, x^2)ds^b.$$

Expressions for this group are given for the cases: the affine torsion $t^a{}_{bc}\neq0$; $t^a{}_{bc}=0$, $t^0{}_{12}\neq0$; $t^a{}_{bc}=t^0{}_{12}=0$. If, in all these cases, the group is the identity, the P_2 admits a group of automorphisms depending on at most two parameters, if the bilinear covariants $w^a{}_{bc}$ and the coefficients of connection $\gamma^a{}_{bc}$ belonging to the Pfaffians are constants.

D. J. Struik (Cambridge, Mass.).

Bell, P. O. Projective Frenet formulas for an analytic curve in n-dimensional space. Univ. Nac. Tucumán. Rev. Ser. A. 10, 83–93 (1954).

Etant donné une variété analytique V_m à m dimensions dans un espace projectif S_n à n dimensions, on peut l'étudier en associant au point générique x_0 de V_m un repère local de référence $x_0x_1\cdots x_n$. Les coordonnées homogènes $x_j^{(p)}$ des points x_j satisfont à un système d'équations de la forme

$$(1)\qquad \partial x_i/\partial u^\alpha=\Gamma_{i\alpha}{}^h x_h \quad (i=0, \cdots, n;\ \alpha=1, \cdots, m).$$

On désigne par x^h les coordonnées locales d'un point variable X de V_m au voisinage du point x_0, par $z^j=x^j/x^0$ les coordonnées non homogènes. La variété V_m est définie au voisinage de x_0 par des développements en série

$$z^r=p^r(z^1, z^2, \cdots, z^m)\quad (r=m+1, \cdots, n).$$

Un problème important de la géométrie projective différentielle d'une variété analytique V_m est de choisir le repère local de façon que ces développements en série soient aussi simples que possible. L'article analysé est consacré à l'étude de ce problème pour une courbe V_1 dans S_n. La normalization adoptée conduit pour $n=3$ au tétraèdre canonique de Halphen; les relations (1) correspondantes sont appelées les formules de Frenet projectives pour la courbe; le point unitaire est caractérisé géométriquement au moyen de l'élément d'arc projectif $d\sigma$. *M. Decuyper* (Lille).

Mihăileanu, N. Une méthode générale d'obtention des formules de Frenet dans les espaces non euclidiens. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 5, 493–502 (1953). (Romanian. Russian and French summaries)

Derivation of the Frenet formulas for a space with metric determined by the form $hx_0^2+x_1^2+\cdots+x_n^2=0$, where $x_0, x_1, \cdots, x_n$ are homogeneous coordinates.

D. J. Struik (Cambridge, Mass.).

Grindei, Ion. Géométrie du groupe des affinités qui laisse invariante la forme différentielle extérieure canonique. II. Théorie des hypersurfaces. Acad. Repub. Pop. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Ser. I. 5, no. 3–4, 85–97 (1954). (Romanian. Russian and French summaries)

A hypersurface V_3 is given in the form

$$x_4=f(x_1, x_2, x_3)=\varphi_0+\varphi_1+\tfrac{1}{2}\varphi_2+\cdots+\frac{1}{n!}\varphi_n+\cdots,$$

$$\varphi_0=A_{000},\quad \varphi_1=A_{100}x_1+A_{010}x_2+A_{001}x_3,\ \cdots.$$

This V_3 is supposed to leave invariant the exterior differential form $[dx_1dx_2]+[dx_3dx_4]$. The equation of the V_3 can then be cast into the form

$$x_4=\tfrac{1}{2}(x_3^2+2mx_1x_2)+\frac{1}{3!}(x_1^3+\alpha x_2^3+\beta x_3^3+3\gamma x_1^2x_2+3\delta x_1^2x_3+3\epsilon x_2^2x_3+3\mu x_2x_3^2+3\nu x_1x_2^2+3\rho x_1x_3^2)+\frac{1}{4!}\varphi_4+\cdots,$$

where $m, \alpha, \beta, \cdots, \rho$ are fundamental invariants. The complete set of these invariants with the integrability relations is derived, after which it is found that the hypersurface given by

$$x_1=\frac{V^3}{6}F_1-\frac{U^3}{6}F_2,\quad x_2=\frac{V^2}{2}F_1+\tfrac{1}{2}U^2F_2,\quad x_3=VF_1-UF_2,$$

$$x_4=F_1+F_2,\quad F_1=U'^{3/2}e^w(U+V)^{-3},\quad F_2=\tfrac{1}{3}e^{-w}U'^{3/2},$$

where $U=U(u)$, $V=V(v)$ and u, v, w are parameters, has constant invariants. These hypersurfaces are affine hyperspheres. The case of curves which leave the given exterior differential form invariant has been taken up in a previous (as yet unavailable) paper. *D. J. Struik.*

***Raševskiĭ, P. K. Rimanova geometriya i tenzornyĭ analiz. [Riemannian geometry and tensor analysis.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 635 pp. 21.70 rubles.

According to the author's introduction this book is intended to be more of a text book than a monograph or a treatise and is intended for students in the "third university course". In the reviewer's opinion the author succeeds admirably in his objectives and goes considerably beyond them. The main characteristics of a text book should be clarity of exposition rigorous presentation and breadth of scope, characteristics fully possessed by the book under review. The first three chapters are devoted primarily to tensor algebra in affine and euclidean and pseudo-euclidean spaces. The treatment is mostly axiomatic and as soon as possible the author applies the acquired information to simple problems of mechanics, elasticity and hydrodynamics. The last half of the third chapter is devoted to spinor analysis and its representation in an affine space. Chapter IV is devoted to the special theory of relativity and includes a fairly complete treatment of the Lorentz transformation, Maxwell's equations for an electromagnetic field and the wave equation of Dirac for a free electron. Chapters V and IX inclusively treat Riemannian geometry proper although here some topics are missing, such as the group of motions of a Riemann space or the whole imbedding theory. The last chapter deals with the general theory of relativity and discusses quite fully the theory of the gravitational field, rotation of planetary orbits and the red shift of spectral lines. There are a few faults, not of a mathematical nature, that bothered the reviewer. Gauss' integral theorem is called Ostrogradsky's theorem; Bianchi's theorem (dealing with the covariant derivative of the curvature tensor) is called the Bianchi-Padova theorem and parts of the brief philosophical discussions in the chapter on general relativity may not be acceptable to non-believers in dialectic materialism. There is no bibliography and in all the 635 pages there are only three or four references to other authors, very recent and Russian. The general excellence of the book more than compensates these minor faults. *M. S. Knebelman.*

Lemoine, Simone. Réductibilité de variétés riemanniennes complètes dans l'espace euclidien. C. R. Acad. Sci. Paris **240**, 1962–1964 (1955).

If V_{n-1} is an $(n-1)$-dimensional Riemannian space imbedded in an n-dimensional Euclidean space so that the induced metric is reducible, then this metric decomposes into two parts, one of which is irreducible and the other is flat. This local result is then used to prove the following global theorem. Let V_{n-1} be a connected $(n-1)$-dimensional Riemannian manifold of class C^3 which is imbedded in n-dimensional Euclidean space so that it is reducible and complete with respect to the induced metric. Then V_{n-1} is the product of $(n-p)$ lines with a connected $(p-1)$-dimensional Riemannian manifold V_{p-1} which is irreducible, complete and imbedded in p-dimensional Euclidean space.

T. J. Willmore (Liverpool).

Ishihara, Shigeru. Fibred Riemannian spaces with isometric parallel fibres. Tôhoku Math. J. (2) **6**, 243–252 (1954).

This paper continues the study of Riemannian manifolds with isometric parallel fibres initiated by Y. Mutō [Sci. Rep. Yokohama Nat. Univ. Sect. I. **1**, 1–14 (1952); MR **15**, 827], and uses results given by the reviewer [Proc. London Math. Soc. (3) **3**, 1–19 (1953); MR **15**, 159]. It includes necessary and sufficient conditions on the bundle structure of a fibred differentiable manifold for it to have isometric parallel fibres.

A. G. Walker (Liverpool).

Legrand, Gilles. Sur les espaces homogènes presque complexes. C. R. Acad. Sci. Paris **240**, 2044–2046 (1955).

A duplication of Theorems 1 to 4 of Lichnerowicz, Arch. Math. **5**, 207–215 (1954) [MR **16**, 520] with "almost complex" replacing "complex".

W. Ambrose.

Legrand, Gilles. Connexions infinitésimales définies sur l'espace fibré des repères affines d'une variété différentiable. C. R. Acad. Sci. Paris **240**, 586–588 (1955).

Let V be a real manifold, $E(V)$ the bundle of bases over V, and $\mathcal{E}(V)$ the bundle of affine bases over V, an affine base at $v \in V$ being any element of the tangent space at v together with any base of that tangent space. There is a natural mapping of $\mathcal{E}(V)$ onto a sub-bundle of $E(V\times R)$ where R is the real line. Every connexion on $\mathcal{E}(V)$ decomposes into a connexion on $E(V)$ and a 1-1 tensor on V. Formulas are given relating the 1-1 tensor and the curvature forms of these connexions.

W. Ambrose (Cambridge, Mass.).

Nožička, Frant. On the problem of the affine normal and the induced connection in a hypersurface in affine space. Časopis Pěst. Mat. **79**, 101–134 (1954). (Czech)

Le travail présenté est le complément d'un mémoire précédent de l'auteur [même Časopis **75**, 179–209 (1950); MR **13**, 72]. Dans l'espace affin à n $(n>2)$ dimensions avec les coordonnées ξ^α $(\alpha=1,\cdots,n)$ et la connexion symmétrique $\Gamma^\nu_{\lambda\mu}$ l'auteur étudie l'hypersurface X_{n-1} donnée par les équations $\xi^\alpha=\xi^\alpha(\eta^1,\cdots,\eta^{n-1})$. Si $B_a{}^\nu=\partial\xi^\nu/\partial\eta^a$, $a=1,\cdots,n-1$, on a, pour le vecteur tangent t_ν, $B_a{}^\nu t_\nu=0$. Le tenseur $h_{ab}=B_a{}^\nu\nabla_b t_\nu$ est important pour la théorie de X_{n-1}. L'auteur suppose $h<n-1$, où h est le rang du tenseur h_{ab}. (Le cas $h=n-1$ est mentionné dans le travail cité.)

Le tenseur l^{ac} pour lequel

$$l^{ac}h_{cj}=\delta_j{}^a,\quad j=n-h,\cdots,n-1;\quad a,c=1,\cdots,n-1,$$

donne avec t_ν et $B_a{}^\nu$ les tenseurs

$$L^c{}_{ab}=l^{cd}\nabla_d t_\nu\nabla_a B_b{}^\nu,\quad M_a=\frac{2}{h+2}([{}_a{}^c{}_c]-L^c{}_{ac}),$$

où

$$[{}_a{}^c{}_b]=\tfrac{1}{2}l^{cd}(\partial_a h_{db}+\partial_b h_{ad}-\partial_d h_{ab}).$$

Le vecteur affinonormal n^ν de la variété X_{n-1} avec le tenseur h_{ab} au rang $1\leqq h<n-1$ est défini par les équations

$$\begin{aligned}&n^\nu t_\nu=1,\\ \text{(a)}\qquad &n^\nu\nabla_j t_\nu=M_j,\quad j=n-h,\cdots,n-1,\\ &n^\nu\nabla_i t_\nu=\sum_{j=n-h}^{n-1}\overset{j}{\underset{i}{\lambda}}M_j+u_i,\quad i=1,\cdots,n-h-1,\end{aligned}$$

où $\overset{j}{\underset{i}{\lambda}}$, u_i sont les grandeurs, qui satisfont aux équations

$$\nabla_i t_\nu=\sum_{j=n-h}^{n-1}\overset{j}{\underset{i}{\lambda}}\nabla_j t_\nu+u_i t_\nu,\quad i=1,\cdots,n-h-1.$$

L'auteur prouve le théorème suivant: Soit donnée dans l'espace affin A_n $(n>2)$ l'hypersurface régulière X_{n-1} et soit, pour le rang du tenseur h_{ab}, $1\leqq h<n-1$ dans tous les points considérés. Soit $\underset{(i)}{v^a}$, $i=1,\cdots,n-h-1$ les solutions linéairement indépendantes des équations

$$\text{(b)}\qquad v^a h_{ab}=0,\quad b=1,\cdots,n-1.$$

Le $(n-h)$-vecteur avec les composantes

$$\underset{(1)}{v^{[\alpha_1}}\,\underset{(2)}{v^{\alpha_2}}\cdots\underset{(n-h-1)}{v^{\alpha_{n-h-1}}}n^{\alpha_{n-h}]},\quad \underset{(i)}{v^\alpha}=B_a{}^\alpha\underset{(i)}{v^a},$$

où n^ν est la solution quelconque des équations (a), définit dans chaque point de la variété X_{n-1} la $(n-h)$-direction qui a les propriétées suivantes: a) Elle ne dépend pas au choix des solutions linéairement indépendantes $\underset{i}{v^a}$ $(i=1,\cdots,n-h-1)$; b) elle ne dépend pas au choix de la solution n^α du système (a); c) elle ne dépend pas au choix du facteur de t_ν. Le $(n-h)$-vecteur défini est la direction invariante affinonormale de la variété X_{n-1}.

Les grandeurs $\underset{1}{B_\nu{}^a}$, qui satisfont aux équations

$$\underset{1}{B_\nu{}^a}B_b{}^\nu=\delta_b{}^a,\quad \underset{1}{B_\nu{}^a}n^\nu=0$$

définissent la connexion

$$\underset{1}{\Gamma^c{}_{ab}}=\underset{1}{B_\nu{}^c}\nabla_a B_b{}^\nu$$

induite dans X_{n-1} par le vecteur afinonormal n^ν.

L'auteur étudie la corespondance entre la connexion et le vecteur n^ν et aussi l'influence du choix du facteur de t_ν et définit pour le vecteur t_ν fixé le vecteur affinonormal n^ν et la connexion invariante.

Pour les hypersurfaces totalement géodétiques, c'est-à-dire pour X_{n-1}, pour lesquelles $h_{ab}\equiv0$, l'auteur définit aussi le vecteur n^ν à l'aide des équations

$$\begin{aligned}\text{(c)}\qquad &n^\nu t_\nu=0,\\ &n^\nu\nabla_a t_\nu=u_a,\quad a=1,\cdots,n-1\end{aligned}$$

où u_a est le vecteur de X_{n-1}, qui satisfait à l'équation $\nabla_a t_\nu=u_a t_\nu$ au chaque point de X_{n-1}.

Si n^ν est une solution des équations (c) et X_{n-1} est la hypersurface régulière avec le tenseur $h_{ab}\equiv0$, on a

$$\underset{1}{\Lambda^c{}_{ab}}=\underset{1}{B_\nu{}^c}\nabla_a B_b{}^\nu,$$

où $\underset{1}{B_\nu{}^c}$ satisfont au système

$$\underset{1}{B_\nu{}^a}B_b{}^\nu=\delta_b{}^a,\quad \underset{1}{B_\nu{}^a}n^\nu=0,$$

la connexion indépendente au choix de la solution n^r du système (c) et aussi au choix du facteur de t_r.

Dans la seconde partie du travail l'auteur montre l'interprétation géométrique de la normale affine définie pour les quadriques dans l'espace euclidien. *F. Vyčichlo.*

Hano, Jun-ichi, and Morimoto, Akihiko. Note on the group of affine transformations of an affinely connected manifold. Nagoya Math. J. **8**, 71–81 (1955).

Nomizu has proved that the group of all affine transformations of a complete affinely connected manifold M is (under the compact-open topology) a Lie group [Proc. Amer. Math. Soc. **4**, 816–823 (1953); MR **15**, 468]. The present authors prove this theorem without the assumption of completeness of M. This is a generalization of the result of the reviewer and Steenrod for isometries of a Riemannian manifold [Ann. of Math. (2) **40**, 400–416 (1939)]. The proof is accomplished by use of a Riemannian metric induced in the bundle of frames of M. *S. B. Myers.*

Vrânceanu, Gheorghe. Sur les espaces à connexion affine partiellement projectifs. Czechoslovak Math. J. **4**(79), 283–286 (1954). (Russian summary)

An affine A_n is said to be partially projective of order $n-p$ if its self-parallel curves given by

$$\frac{d^2x^i}{dt^2}=\Gamma^i{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}$$

can be expressed by $n-p$ linear equations and $p-1$ equations which need not be linear, $p>1$. If $p=1$ the A_n is projective euclidean. It is shown that, if the A_n has in every hyperplane the maximum number ∞^{2n-4} of self-parallel curves, it is projective euclidean (and conversely); if it has this maximum number in every hyperplane of a family of ∞^{n-p} parallel hyperplanes, the A_n is partially projective of order $n-p$ (and conversely). It is possible to get partially projective spaces of a more general nature by taking, for instance, the case that the hyperplane $x^1=c^1$ contains ∞^{2n-5} self-parallel curves. Then

$$\frac{\partial\Gamma^1{}_{\alpha\beta}}{\partial x^\gamma}+\frac{\partial\Gamma^1{}_{\beta\gamma}}{\partial x^\alpha}+\frac{\partial\Gamma^1{}_{\gamma\alpha}}{\partial x^\beta}+\Gamma^1{}_{\alpha\rho}\Gamma^\rho{}_{\beta\gamma}+\Gamma^1{}_{\beta\rho}\Gamma^\rho{}_{\gamma\alpha}+\Gamma^1{}_{\gamma\rho}\Gamma^\rho{}_{\alpha\beta}=0.$$

D. J. Struik (Cambridge, Mass.).

Schouten, J. A., and Yano, K. On the geometrical meaning of the vanishing of the Nijenhuis tensor in an X_{2n} with an almost complex structure. Nederl. Akad. Wetensch. Proc. Ser. A. **58** = Indag. Math. **17**, 133–138 (1955).

At every point of a manifold with an almost complex structure there are a mixed second-order tensor F satisfying $F^2=-I$ and certain vector spaces invariant under the transformation F. The question considered here is whether there are sub-manifolds whose tangent spaces are these invariant spaces, and it is shown that this depends upon the vanishing of the Nijenhuis tensor $N_{ji}{}^h$ of F. If the manifold is analytic the invariant space at any one point is arbitrary. It is also shown that the vanishing of the Nijenhuis tensor implies the integrability of a more general system of equations, and the geometrical interpretation of these equations is given. *A. G. Walker* (Liverpool).

Kobayashi, Shôshichi. Espaces à connexion de Cartan complets. Proc. Japan Acad. **30**, 709–710 (1954).

Let E be a fiber bundle with a Cartan connection and H the associated principal bundle of E. Let ω be the differential form which defines the Cartan connection. It is proved that, in order that E be complete, it is necessary and sufficient that every vector field v in H such that $\omega(v)$ is a constant generates a global one-parameter group of transformations. This theorem is applied to give a proof of the theorem that if the Cartan connections of two simply-connected spaces are analytic and complete, then every local isomorphism can be extended to a global isomorphism. Another consequence of the theorem is the following: On a complete Riemannian manifold every Killing vector field generates a global one-parameter group of isometries. *S. Chern.*

Takasu, Tsurusaburo. Connection spaces in the large. X. A new view on the relation between the "Erlanger Programm" and the linear connections. Yokohama Math. J. **2**, 81–94 (1954).

The purpose of this paper is to add a new view to the relation between the "Erlanger Programm" and the linear connections from the point of view of the author's "II-geodesic curves", which lie entirely in the differentiable manifold in consideration and behave for meet and join like ordinary straight lines. A fundamental principle is given at the end. *J. A. Schouten* (Epe).

Blaschke, Wilhelm. Zur Variationsrechnung. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **19**, 106–107 (1954). (Turkish summary)

The problem is to find those Finsler metrics in a 2-space such that all Finsler circles (defined by constancy of Finsler radius) are also Euclidean circles. Or, in optical language, to find those refractive indices for which all waves from all point sources are circles [cf. B. Segre, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **7**, 16–19, 20–26 (1949); MR **11**, 742]. The author outlines a method of solution based on contact transformations and the group of conformal transformations of a 3-space of circles with metric $ds^2=da_1^2+da_2^2-db^2$, where (a_1, a_2) is the centre of a circle and b its radius. *J. L. Synge* (Dublin).

Haimovici, A. On the notion of tensor. Gaz. Mat. Fiz. Ser. A. **7**, 62–78 (1955). (Romanian)

NUMERICAL AND GRAPHICAL METHODS

Householder, Alston S. Bibliography on numerical analysis. Oak Ridge National Laboratory, Oak Ridge, Tenn., Rep. **ORNL 1897**, 32 pp. (1955).

A list of 321 papers or books chiefly concerned with "algebraic numerical analysis" and supplementary to the one in the author's Principles of numerical analysis [McGraw-Hill, New York, 1953; MR **15**, 470].

Kaye, Joseph. A table of the first eleven repeated integrals of the error function. J. Math. Phys. **34**, 119–125 (1955).

The error function is defined by

$$\operatorname{erf} x=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}d\xi$$

and its repeated integrals by

$$\mathrm{i}^n \operatorname{erfc} x = \int_x^\infty \mathrm{i}^{n-1} \operatorname{erfc} \xi \, d\xi,$$

$$\operatorname{erfc} x = 1 - \operatorname{erf} x, \quad \mathrm{i}^0 \operatorname{erfc} x = \operatorname{erfc} x.$$

The tables give values of $\mathrm{i}^n \operatorname{erfc} x$ for $n=0(1)11$ and $x=0(0.01)0.25(0.05)1.00(0.01)3.00$, or until the function vanishes to the number of figures given. This is such that the entries for $x=0$ have six significant figures for $n=0(1)7$, five significant figures for $n=8(1)11$. Second differences are provided, often but not always adequate for accurate interpolation. Computation was effected by use of a simple recurrence relation for fixed x and varying n, with enough guarding figures to absorb the loss of significant figures, and checked by differencing in the x direction. The tables were prepared to assist the design of charts for rapid solution of heat flow problems at supersonic speeds. *L. Fox.*

Vaughan, Hubert. Osculation of high order. J. Inst. Actuar. 81, 53–57 (1955).

✱Schmitt, A. Détermination de la fonction homographique à partir de données expérimentales en surnombre, les abscisses étant en progression arithmétique. L'exploitation des données empiriques. Publ. Sci. Tech. Ministère de l'Air, Paris, Notes Tech. no. 52, pp. 41–53 (1955). 850 francs.

The equation being fitted is $(x+a)(y+b)=c$. The empirical quantities are substituted and the resulting approximate relations combined in certain ways by utilizing Cracovian algebra to give expressions for a, b and c. There is no explicit statement of a criterion of best fit.
A. S. Householder (Oak Ridge, Tenn.).

✱Schmitt, A. Représentation d'une série expérimentale par une fonction de la forme $y=a+be^{ax}$. L'exploitation des données empiriques. Publ. Sci. Tech. Ministère de l'Air, Paris, Notes Tech. no. 52, pp. 55–59 (1955). 850 francs.

The treatment here is in the same spirit as for the homographic function discussed in the preceding review.
A. S. Householder (Oak Ridge, Tenn.).

Booth, A. D. A note on approximating polynomials for trigonometric functions. Math. Tables Aids Comput. 9, 21–23 (1955).

For use with high-speed digital computers, series expansions are given for $\sin(\pi x/2)$ and $\cos(\pi x/2)$, both in terms of Chebyshev polynomials and in terms of Legendre polynomials. The former are recommended when full machine accuracy is needed, the latter in other cases. The coefficients of the expansions involve Bessel functions $J_n(\pi/2)$ and $J_{n+\frac{1}{2}}(\pi/2)$, and eleven-decimal values of these quantities are tabulated for $n=0(1)12$. *L. Fox* (Teddington).

Atta, Susie E., and Sangren, Ward C. Calculation of generalized hypergeometric series. J. Assoc. Comput. Mach. 1, 170–172 (1954).

A code for the calculation, on an automatic computer, of ${}_pF_q(\alpha_1, \alpha_2, \cdots, \alpha_p; \rho_1, \rho_2, \cdots, \rho_q; x)$ is described. Scaling and floating point operations are incorporated; the series is truncated when the absolute value of the terms become small relative to the accumulated partial sum. See also the paper reviewed below. *John Todd* (Washington, D. C.).

Henrici, Peter. Application of two methods of numerical analysis to the computation of the reflected radiation of a point source. J. Washington Acad. Sci. 45, 38–45 (1955).

The problem is to evaluate numerically the double integral $\int_0^\infty dt \int_0^{2\pi} tu^{-3}v^{-2}d\phi$, where $u^2=1+\xi^2+t^2-2t\cos\phi$ and $v=t^2+\eta^2$. The integral is first transformed into a single integral involving a hypergeometric function. The hypergeometric function is calculated from its power series expansion and the integral evaluated numerically. Most of the paper is concerned with estimates of the errors produced by truncation of the power series and by the quadrature. In order to speed up the convergence of the series, Aitken's δ^2-method is studied in detail. For the quadrature errors, complex variable methods recently developed by Davis and Rabinowitz [Math. Tables Aids Comput. 8, 193–205 (1954); MR 16, 404] are employed. It is reported that values of the original integral for the two ranges $\xi=.05(.05)1.6$, $\eta=.05(.05)1.6$ and $\xi=.25(.25)8.0$, $\eta=.25(.25)8.0$ have been calculated and are kept on file. *P. W. Ketchum.*

Aitken, A. C. Note on the acceleration of Lin's process of iterated penultimate remainder. Quart. J. Mech. Appl. Math. 8, 251–255 (1955).

Lin [J. Math. Phys. 20, 231–242 (1941); MR 3, 153] introduced the method of iterated penultimate remainder for the approximation of the factors of a polynomial. Here the method of the catalytic multiplier is described, that is, by multiplication of the polynomial by a suitably determined catalytic polynomial, divergence is removed and convergence accelerated. The method is illustrated for a linear and a quadratic factor. *E. Frank.*

Hammer, Preston C. Iterative procedures for taking roots based on square roots. Math. Tables Aids Comput. 9, 68 (1955).

Dantzig, George B., Orden, Alex, and Wolfe, Philip. The generalized simplex method for minimizing a linear form under linear inequality restraints. Pacific J. Math. 5, 183–195 (1955).

This paper, which has circulated privately for some time as a RAND Corporation publication, presents a readable account of the well-known method conceived by Dantzig for solving linear programs. The principal element of novelty is the imbedding of the problem into a generalized matrix problem, which has as a by-product another resolution of the degeneracy difficulty. *A. J. Hoffman.*

Forsythe, G. E., and Straus, E. G. On best conditioned matrices. Proc. Amer. Math. Soc. 6, 340–345 (1955).

The condition number $P(A)$ of a positive definite hermitian matrix A is the ratio of the maximal and minimal eigenvalue Λ/λ. If $P(A)$ is small the error in the numerical inversion (by a Gaussian elimination process) of A will be small too [see V. Neumann and Goldstine, Bull. Amer. Math. Soc. 53, 1021–1099 (1947); MR 9, 471; see also J. Todd, Arch. Math. 5, 249–257 (1954); MR 16, 523]. The authors define A to be "best conditioned" with respect to a set of regular linear transformations $\{T\}$ if $P(T^*AT) \geq P(A)$ for all T. Let x, y denote unit eigenvectors belonging to Λ, λ respectively. Assume that no constant k can be found such that for some T and all x, y we have either

$$x^*(T^*)^{-1}T^{-1}x < k < y^*(T^*)^{-1}T^{-1}y$$

or

$$x^*(T^*)^{-1}T^{-1}x > k > y^*(T^*)^{-1}T^{-1}y.$$

It is shown that in this case A is best conditioned with respect to any set containing T. The converse is, in general, not true, but does hold for "infinitesimally complete" sets $\{T\}$, i.e. sets for which to every T arbitrarily small positive ϵ, ϵ' and $T_\epsilon, T_{\epsilon'}$ exist such that $I+\epsilon(T^*)^{-1}T^{-1}=c(T_\epsilon^*)^{-1}T_\epsilon^{-1}$, $I-\epsilon'(T^*)^{-1}T^{-1}=c'(T^*_{\epsilon'})^{-1}T_{\epsilon'}^{-1}$, where c, c' are positive numbers. Examples of such a class are the matrices which have square blocks along the diagonal and zero elsewhere. The ordinary diagonal matrices D form a subclass and the process T^*AT is then simply scaling of rows and columns. A sufficient condition (also necessary if Λ, λ are both simple) for A to be best conditioned with respect to the class $\{D\}$ is that for some pair of eigenvectors belonging to Λ and λ the absolute values of the coordinates coincide. Using this result the following conjecture of D. M. Young concerning an iterative solution of certain systems of linear equations is established: If A is of the form $\begin{pmatrix} I_p & B \\ B^* & I_q \end{pmatrix}$, where I_p, I_q are unit matrices, then it is best conditioned with respect to $\{D\}$. Some further remarks and conjectures concerning this special A are given. *O. Taussky-Todd.*

Salzer, Herbert E. Equally weighted quadrature formulas over semi-infinite and infinite intervals. J. Math. Phys. 34, 54–63 (1955).

Quadrature formulas of the form $\sum c_i f(x_i)$ with equal c_i's for $\int F(x)f(x)\,dx$ are studied for cases where $F(x)$ is either $\exp(-x)$ or $\exp(-x^2)$ and the limits of integration are 0 to ∞ and $-\infty$ to ∞, respectively. For n-point formulas which are exact for polynomials of degree n the x_i's are zeros of polynomials whose coefficients are tabulated for $n=2, \cdots, 10$. Some of the x_i's are imaginary (values of which are tabulated) for $2<n\leqq 10$ and $3<n\leqq 10$ on the semi-infinite and infinite ranges, respectively, and it is conjectured that the same is true for every $n>10$. If the formula is only required to be exact for polynomials of degree $m<n$, then additional cases exist with all x_i's in the range of integration. All such cases are determined for $n\leqq 14$ and $n\leqq 21$ for the semi-infinite and infinite ranges, respectively. *P. W. Ketchum* (Urbana, Ill.).

Gurk, Herbert M. The use of stability charts in the synthesis of numerical quadrature formulae. Quart. Appl. Math. 13, 73–78 (1955).

The paper is related to real-time simulation problems. It continues an investigation by H. J. Gray [same Quart. 12, 133–140 (1954); MR 15, 991]. Gray considered a set of ordinary differential equations $dx/dt=Ax$ with A as a constant $n\times n$ matrix and with x as a column-vector. If A admits n different eigenvalues λ_k, then the general solution is of course $x=\sum_{k=1}^n \exp(\lambda_k t)f_k$ with the vectors f_k being eigenvectors to A and λ_k. Real-time simulation requires one to find an approximation y to x by means of a difference method, the latter being related to the grid $t=kh$ with k as integer and h as a constant width of step. Such a difference method will usually be represented by a set of difference equations with constant coefficients. Hence it leads to $y=\sum_{k=1}^n \exp(\mu_k t)g_k$ with suitable vectors g_k for the grid points $t=ih$. The values μ_k depend on A and on the method. In general there is a relation $h\lambda_k=\phi(h\mu_k)$ with $z=\phi(w)$ as analytic function. For reasons of stability the real parts of the numbers λ_k, μ_k and especially $\max_k \mathrm{Re}\lambda_k$ and $\max_k \mathrm{Re}\mu_k$ as well as the relation between them deserve special interest.

While Gray gave special attention to the conformal mapping $z=\phi(w)$ for given methods, the author aims at a synthesis of new difference formulae satisfying $\phi(0)=0$ and $\phi(w)\approx w$ in as large a neighborhood of the origin as possible. In this connection the case $\phi^{(r)}(0)=0$ for $r=0, 2, 3, \cdots, m$; $\phi'(0)=1$ is discussed. Some points remained unclear to the reviewer. *H. Bückner* (Schenectady, N. Y.).

Jacchia, Luigi. On the numerical integration of functions tabulated in logarithmic form. Math. Tables Aids Comput. 9, 63–65 (1955).

Dengler, Max. Numerische Lösung des Integrals

$$\int_C f(u)/\{w(u)-w(q)\}^2du.$$

Z. Angew. Math. Mech. 34, 471–474 (1954).

Integrals of the form $\int f(u)/\{w(u)-w(q)\}^2du$ are approximated for contours passing near $u=q$, using series expansions. Numerical examples are worked out. *G. Birkhoff* (Cambridge, Mass.).

***Puig-Adam, P. Transformées de Laplace des fonctions empiriquement données.** Les machines à calculer et la pensée humaine, pp. 263–278. Colloques internationaux du Centre National de la Recherche Scientifique, no. 37. Centre National de la Recherche Scientifique, Paris, 1953. 2000 francs.

A survey is made of various means for calculating the Laplace transform of $f(x)$ numerically or mechanically from experimentally or otherwise given values. The integration limits 0 to ∞ are truncated to 0 to L and an estimate of the error is given. It is assumed that on 0 to L the function $f(x)$ can be approximated uniformly by a polynomial $f_n(x)$ in such a way that a similar truncation of the transform of $f_n(x)$ also involves only a small error. Then the transform $F_n(p)$ of $f_n(x)$ is a polynomial in $1/p$ which will approximate the transform of $f(x)$. In addition to approximation by polynomials, the author discusses approximation of $f(x)$ by exponential polynomials, mechanical evaluation of the transform as a Stieltjes integral, and also an iterative procedure obtained by expanding the exponential factor of the integrand in a Taylor's series. *P. W. Ketchum.*

Fox, L. A note on the numerical integration of first-order differential equations. Quart. J. Mech. Appl. Math. 7, 367–378 (1954).

The author points out the difficulty of replacing a first-order differential equation by a suitable difference equation: if the difference equation is also of first order it lacks both symmetry and accuracy, and if it is symmetrical it is of second order and cannot be solved without one additional item of data. He then shows the advantages of replacing the first-order differential equation by an equivalent second-order differential equation with the first derivative eliminated. This leads to a simple second-order difference equation, which is generally a good representation of the differential equation, and which is readily solved either step-by-step or by relaxation. *W. E. Milne.*

Bishop, R. E. D. On the graphical solution of transient vibration problems. Proc. Inst. Mech. Engrs. 168, 299–312; discussion 312–322 (1954).

This is a lucid expository article thoroughly developing some graphical techniques for solving second-order ordinary differential equations. Step-by-step devices based upon the geometric interpretations of the equation quantities as represented in the phase plane are used. Viscous damping and some nonlinearity may be introduced with moderate additional complication. The method is adapted to transient

rather than asymptotic vibrations. The idea is extended to systems of more than one degree of freedom by transforming to normal modes. Systems of infinitely many degrees of freedom may be treated by considering the coefficients in a Fourier expansion. The application of the method to several important classes of elastic problems is discussed. Jacobsen's graphical procedure for a more general class of differential equations is outlined. The extensive appended discussion of the paper by various contributors emphasizes its importance in undergraduate instruction. *E. Pinney.*

Vladimirov, V. S. Approximate solution of a boundary problem for a differential equation of second order. Prikl. Mat. Meh. 19, 315–324 (1955). (Russian)

The equation is of the form $y''-py=-f$ with boundary conditions $y'(0)-H_1y(0)=y'(1)+H_2y(1)=0$ where H_1 and H_2 are non-negative. The author uses the factorization $(D^2-p)=(D-g)(D+g)$, where $D=d/dx$ and g satisfies a Riccati equation. The numerical solution is built up by replacing p and f by step functions, each constant value being an integral mean, and solving the simplified problem over the subintervals. Assuming only summability of p and f, the author proves uniform convergence of the approximate solution and its first derivative, and convergence in the mean for the second derivative.

A. S. Householder (Oak Ridge, Tenn.).

Lyusternik, L. A. Application of cubature formulas to the numerical solution of Cauchy's problem for certain equations of mathematical physics. Vyčisl. Mat. Vyčisl. Tehn. 1, 14–26 (1953). (Russian)

Consider the differential equation

$$A(d)u(x,y,t)=B(d_1,d_2)u(x,y,t) \tag{1}$$

in which $d=\partial/\partial t$, $d_1=\partial/\partial x$, $d_2=\partial/\partial y$, $A(d)$ is a polynomial in d of degree n, and $B(d_1,d_2)$ is a polynomial in the operators d_1 and d_2. Solutions of (1) are sought which satisfy initial conditions

$$\left.\frac{\partial^k u}{\partial t^k}\right|_{t=0}=\varphi_k(x,y) \quad (k=0,1,\cdots,n-1). \tag{2}$$

Let (x_i,y_i) $(i=0,1,\cdots,n-1)$, denote the coordinates of n equally spaced points on a circle of radius ρ with center at the origin. Then by means of the translation operator $\exp\{x_id_1+y_id_2\}$ it is possible to express a sum of the type

$$\sum_{i=0}^{n-1}\psi(x+x_i,y+y_i) \tag{3}$$

as a result of an operator $S(\rho,d_1,d_2)$ operating on the function $\psi(x,y)$. By different choices of ρ several such operators may be obtained, $S_1,S_2,\cdots,S_m$. Now (1) can be solved formally in terms of t and the result expressed thus:

$$u=\sum_{k=0}^{n-1}F_k(d_1,d_2;t)\varphi_k(x,y). \tag{4}$$

The heart of the method is now to express the operators $F_k(d_1,d_2;t)$ by means of $S_1,S_2,\cdots,S_m$, and hence finally the right-hand member of (4) is expressed by sums of the type (3) applied to $\varphi_k(x,y)$.

Several examples are worked in detail. *W. E. Milne.*

Ivanova, A. I. Certain cases of L. A. Lyusternik's cubature formula for regular polygons. Vyčisl. Mat. Vyčisl. Tehn. 1, 27–36 (1953). (Russian)

The method described in the preceding review is applied to a number of specific cases, which are worked out in full. These include the case of a triangle, a square, a hexagon, a right hexagonal prism, and a cube. *W. E. Milne.*

Saul'ev, V. K. On the solution of the problem of eigenvalues by the method of finite differences. Vyčisl. Mat. Vyčisl. Tehn. 2, 116–144 (1955). (Russian)

For the operator L defined by

$$Lu=\sum_i\partial(a_i\partial u/\partial x_i)/\partial x_i+au,$$

one wishes to satisfy $Lu+\lambda u=0$ in a region Q by a continuous function $\bar{u}$ which vanishes on the boundary of Q. The results are not easily summarized, but the author's purpose is to study convergence of the eigenvalues of lower order, as the mesh decreases in size, for each of three different methods of handling the mesh points on or near the boundary. Rather complicated inequalities are obtained.

A. S. Householder (Oak Ridge, Tenn.).

Andreev, B. A. Calculations of the spatial distribution of potential fields and their application to prospecting geophysics. IV. Izv. Akad. Nauk SSSR. Ser. Geofiz. 1954, 49–64 (1954). (Russian)

[For parts I–III see same Izv. 11, 79–92 (1947); 13, 256–267 (1949); 1952, no. 2, 22–30; MR 11, 108; 14, 92.] The interpretation of magnetic anomaly due to a vertical infinite dike magnetized vertically is the main topic of this paper. The use of the curve $\delta Z=f(x)-\frac{1}{2}[f(x+l)+f(x-l)]$ instead of the observed anomaly $Z=f(x)$ is advised in order to eliminate the disturbing influences such as the regional anomaly, the finite depth of the dike, the contribution of other adjacent anomalies, etc. It is claimed that the depth to the top of a buried dike can be estimated with an accuracy of 10–20%. Application of the same method to the study of depth of basement is sketched. The method can be applied to the gravimetric anomaly of the first vertical derivative $\partial g/\partial z$ of gravity g. *E. Kogbetliantz.*

Harkeevič, Yu. F. Graphical solution of integral equations. Inžen. Sb. 15, 207–215 (1953). (Russian)

In this paper the author gives certain geometrical interpretations of Fredholm and Volterra integral equations from which he develops a method for the graphical solution of these equations. His method carries over to a graphical procedure the familiar analytic method of successive approximations. He employs the procedure of A. B. Štykan [Inžen. Sb. 13, 177–186 (1952); MR 14, 800] for the construction of the integral of the product of two functions.

W. E. Milne (Corvallis, Ore.).

Štykan, A. B. Graphical methods of solution of integral equations. Inžen. Sb. 15, 216–222 (1953). (Russian)

The author presents methods for the graphical solution of nonlinear equations in the form

$$u(x)=f(x)+\int_a^b K(x,s)[u(s)]^m ds$$

and the corresponding form with x in place of b in the upper limit, where in each case the kernel is of special type: $K(x,s)=\sum_{i=1}^n\psi_i(x)\varphi_i(s)$. *W. E. Milne.*

Elste, G. Die Entzerrung von Spektrallinien unter Verwendung von Voigtfunktionen. Z. Astrophys. 33, 39–73 (1953).

A practical method of solving convolution integral equations of the type $S(x)=\int_{-\infty}^{\infty}W(x-y)A(y)\,dy$, for the un-

known function W, is presented. In applications to spectroscopy, the functions here represent intensities for variations in wave length; W is the true intensity profile, S the measured profile and A the profile for the measuring apparatus. The method of solution is graphical and numerical, based on the assumption that all three functions are of the type $f(v)=\int_{-\infty}^{\infty} f_1(v-x)f_2(x)\,dx$, where

$$\pi f_1(x)=\beta_1(\beta_1^2+x^2)^{-1}, \quad \beta_2 f_2(x)=\pi^{-1/2}\exp(-x/\beta_2)^2,$$

and β_1 and β_2 are parameters. This type of function, named after its originator W. Voigt, is introduced for physical reasons, and because the convolution of two such functions is again a function of that type. With the aid of numerical tables involving the parameters in these functions, the author shows how the parameters that determine W can be evaluated in terms of the parameters that fit the functions A and S to their observed values. The method is extended to the representation of A and S as the sum of two functions of the Voigt type. Numerical examples are given.

R. V. Churchill (Ann Arbor, Mich.).

Fischer, Joh. Über eine "nicht nomographierbare" Funktion. Acta Hydrophys. 2, 5–9 (1954).

A function of three variables is said to be nomographable if the function can be represented by three graduated scales whose corresponding valid values lie along a straight line. Formulas in engineering hydraulics are frequently empirical and are, at best, difficult to nomograph. The paper gives several examples of such formulas and indicates how they may be graphed by means of sets of curves coupled with the usual alignment nomograms. As an example, the author selects a "non-nomographable" formula from a book by Pentkovskiĭ [Nomography, Gostehizdat, Moscow-Leningrad, 1949; MR 13, 78; 15, 902] and shows the details of transformation and graphing to give unique solutions. This equation could easily be transformed into a quadratic and nomographed in the usual way but it would then have an extraneous solution. *M. Goldberg* (Washington, D. C.).

Johnston, L. S. The construction and use of nomographic charts. Indust. Math. 3, 69–91 (1952).

Lenaerts, E. H. Automatic square rooting. Electronic Engrg. 27, 287–289 (1955).

Erlee, Th. J. D., and Koene, A. A. The preparation of a system of function punched cards. Statistica, den Haag 8, 155–167 (1954). (Dutch. English summary)

Ericsson, Lars-Eric, Käufl, Josef, and Olsson, Carl Olof. Apparatus for semi-automatic transfer of length-represented data to punched cards and charts. Flygtekn. Försöksanstalt. Rep. 56, 12 pp. (1955).

Fröberg, Carl-Erik. Numerical calculations on digital computers. Nordisk Mat. Tidskr. 3, 33–47, 80 (1955). (Swedish. English summary)

Blanc, Charles. Calcul numérique et calculatrices automatiques. Bull. Tech. Suisse Romande 81, 178–181 (1955).

Huskey, H. D. The influence of automatic computing machines on mathematical research. Indust. Math. 4, 39–48 (1953).

Speedy, C. B. The function of basic elements in digital systems. Proc. Inst. Elec. Engrs. C. 102, 49–56 (1955).

Läuchli, P. ERMETH, le calculateur électronique de l'Ecole Polytechnique Fédérale de Zurich. Bull. Tech. Suisse Romande 81, 182–186 (1955).

Couffignal, M.-L. Le laboratoire de calcul mécanique de l'Institut Blaise Pascal. Bull. Tech. Suisse Romande 81, 181–182 (1955).

Lipton, S. A note on the electronic computer at Rothamsted. Math. Tables Aids Comput. 9, 69–70 (1955).

Loveman, Bernard. "REAC" computer reliability. Tele-Tech 13, no. 3, 79–81, 146, 148, 150 (1954).

Johnson, E. Calvin. Components of digital computers. Indust. Math. 3, 92–103 (1952).

Allen, M. W. A beam-deflection valve for use in digital computing circuits. Proc. Inst. Elec. Engrs. C. 102, 57–61 (1955).

Moisil, Gr. C. Congruences of integers in the theory of automatic mechanisms. Gaz. Mat. Fiz. Ser. A. 7, 53–61 (1955). (Romanian)

Oliwa, Godfried. Zum Wurzelziehen mit der Rechenmaschine. Österreich. Z. Vermessgswes. 43, 54–55 (1955).

Barrios, J. M., and Gonzalez, A. Automatic calculation of the parts of a tetrahedron. Calc. Automat. y Cibernet. 4, no. 10, 20–26 (1955). (Spanish. English summary)

Jacobson, Arvid W. Analogue computation. Indust. Math. 1, 53–63 (1950).

***Raymond, F. H. Conceptions générales d'opérateurs mathématiques électroniques.** Les machines à calculer et la pensée humaine, pp. 185–208; discussion, 208–210. Colloques internationaux du Centre National de la Recherche Scientifique, no. 37. Centre National de la Recherche Scientifique, Paris, 1953. 2000 francs.

Let a system of ordinary differential equations be written in the form $(A+Bp)E=0$, where A and B are constant $n\times n$ matrices, E an unknown vector and p the differential operator with respect to the time t. The author investigates how the preceding system can be solved by means of an electronic analogue computer. To this end he considers a passive linear network with n input poles and n output poles. The input currents, output currents, input potentials and output potentials against ground may be represented by the column vectors I_1, I_2, E_1, E_2 respectively. These are related by the differential equations $I_1=Y_{11}E_1+Y_{12}E_2$, $I_2=Y_{21}E_1+Y_{22}E_2$ with operational matrices Y_{ik}, the elements of which are rational functions of p (if p is replaced by the complex frequency $i\omega$, then the preceding relations may stand for Kirchhoff's laws with the vectors representing amplitudes and the matrices representing admittances). The basic idea is to enforce $E_2=0$, $I_2=0$, which would lead to $Y_{21}E_1=0$; with $Y_{21}=A+Bp$, the latter relations would express the given differential equations. The conditions $E_2=I_2=0$ are practically enforced by n high-gain amplifiers, each with internal gain $g(p)$, input admittance y_e and output impedance z_s. Each amplifier is controlled by one of the components of E_2, and its output is connected to one of the input poles. The effect of all amplifiers is given by $PE_1=g(p)E_2-z_sPI_1$, $I_2=-y_eE_2$ with P as a permutation matrix. In any servo-connected mechanical differential analyser, the differential equations of the servo-mechanisms perturb the differential equations which are actually to be

solved. The same happens here with the amplifiers perturbing the idealized equations $Y_{31}E_1=0$. Therefore the author studies in detail questions of stability and errors, due to the perturbation. These are linked to the characteristic roots of the systems under consideration. The investigation gives numerical details for practical networks. *H. Bückner.*

Nicolau, Edmond. L'étude d'un système différentiel non linéaire par une méthode électronique. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 945–953 (1954). (Romanian. Russian and French summaries)

L'Auteur montre la manière dont on peut étudier le système différentiel non linéaire:

$$dx/dt=ydy/dt=-k^2x+\mu f(x,y,\mu)$$

par une méthode électronique.

From the author's summary.

Elliott, D. R. A location estimator for non-Gaussian distribution. Indust. Math. **1**, 65–70 (1950).

Szerszeń, S. Movable perspectograph. Prace Mat. **1**, 113–130 (1955). (Polish. Russian and English summaries)

Ferro, H. Graph paper. Statistica, den Haag **8**, 123–154 (1954). (Dutch. English summary)

The paper gives a survey of several types of data paper. The designs mentioned are: equidistant paper; ratio paper; logarithmic paper; time or calendar paper; triangular-coordinate paper; polar-coordinate paper; normal-probability paper. Special attention is given to binomial probability paper.

From the author's summary.

RELATIVITY

Kilmister, C. W., and Stephenson, G. Field equations in general relativity. Nuovo Cimento (10) **1**, 361–362 (1955).

Horváth, J. I. Contribution to the final affine field law. Bull. Acad. Polon. Sci. Cl. III. **3**, 151–155 (1955).

The transformation

$$(1)\qquad {}^*\Gamma^{\nu}{}_{\lambda\mu}=\Gamma^{\nu}{}_{\lambda\mu}+2\delta_{\lambda}{}^{\nu}p_{\mu}$$

preserving parallelism yields

$$(2)\qquad (n-1)p_{\mu}=\Gamma_{[\mu}{}^{\lambda}{}_{\lambda]}$$

for

$$(3)\qquad {}^*\Gamma_{[\mu}{}^{\lambda}{}_{\mu]}=0$$

so that one obtains for the corresponding contracted curvature tensors $R_{\mu\lambda}$ and ${}^*R_{\mu\lambda}$ the Schrödinger formula

$$(4)\qquad R_{\lambda\mu}={}^*R_{\lambda\mu}-F_{\lambda\mu}$$

with $F_{\lambda\mu}$ proportional to $\partial_{[\lambda}p_{\mu]}$. In the last section the author expresses the opinion that "the equations of motion in their parameter invariant form must be invariant against changes of the affinities which preserve the parallelism". [Cf. A. Einstein, Louis de Broglie, physicien et penseur, Editions Albin Michel, Paris, 1953, pp. 337–342; MR **16**, 634; and V. Hlavatý, J. Rational Mech. Anal. **3**, 645–689 (1954); MR **16**, 408.] *V. Hlavatý* (Bloomington, Ind.).

Širokov, M. F. On the center of inertia in the general theory of relativity. Ž. Eksper. Teoret. Fiz. **27**, 251–256 (1954). (Russian)

A covariant expression for the coordinate of the center of inertia is set up under suitable assumptions. In the case of slowly moving particles in a weak gravitational field, this agrees with the expression that one obtains by taking into account both the rest mass and the kinetic and potential energy. *N. Rosen* (Haifa).

*****Fok, V. A. The problem of motion of masses in Einstein's theory of gravitation.** Sbornik posvyaščennyĭ semidesyatiletiyu akademika A. F. Ioffe [Collection in honor of the seventieth birthday of academician A. F. Ioffe], pp. 31–43. Izdat. Akad. Nauk SSSR, Moscow, 1950.

This is mainly a discussion of the derivation of the equations of motion of masses from the gravitational field equations along the lines of an earlier work [V. A. Fok, Ž. Eksper. Teoret. Fiz. **9**, 375–410 (1939)]. The masses are described by finite regions of space in which the energy-momentum density tensor differs from zero and are assumed to have spherical symmetry. The calculations are carried out under the assumptions of weak fields and small velocities. A characteristic feature of the work is the imposition of the condition that the coordinates be harmonic. *N. Rosen.*

Kaškarov, V. P. On the equations of motion of a system of finite masses in Einstein's theory of gravitation. Ž. Eksper. Teoret. Fiz. **27**, 563–570 (1954). (Russian)

This is a generalization of the work of V. A. Fok [same Ž. **9**, 375–410 (1939); also see preceding review] on the equations of motion of masses as consequences of the gravitational field equations. The earlier work dealt with bodies having spherical symmetry, whereas the present paper considers bodies of arbitrary shape. *N. Rosen* (Haifa).

Fok, V. A. On the paper of F. I. Frankl', "Some remarks on principles in the general theory of relativity." Uspehi Mat. Nauk (N.S.) **9**, no. 4(62), 229–236 (1954). (Russian)

This is a criticism of an article by F. I. Frankl' [Uspehi Mat. Nauk (N.S.) **8**, no. 3(55), 160–164 (1953); MR **15**, 656] which, among other things, criticized previous work of the author [V. A. Fok, Ž. Eksper. Teoret. Fiz. **9**, 375–410 (1939)]. It deals mainly with questions connected with the solutions of the gravitational field equations, such as the initial conditions, the uniqueness of the solution, and the auxiliary coordinate conditions. *N. Rosen* (Haifa).

Takasu, Tsurusaburo. A necessary unitary field theory as a non-holonomic parabolic Lie geometry realized in the three-dimensional Cartesian space. II. Proc. Japan Acad. **30**, 702–708 (1954).

In a previous paper [same Proc. **29**, 533–536 (1953); MR **16**, 184] Einstein's field theory was brought into connection with a non-holonomic parabolic Lie geometry. In this formalism generalizations are given of the Schrödinger equation and the Dirac equation, both for a single particle. *J. Haantjes* (Leiden).

Belinfante, Frederik J. Use of the flat-space metric in Einstein's curved universe, and the "Swiss-cheese" model of space. Phys. Rev. (2) **98**, 793–800 (1955).

Papapetrou [Proc. Roy. Irish Acad. Sect. A. **52**, 11–23 (1948); MR **10**, 157] has formulated the (non-covariant) De Donder-Lanczos conditions

$$(1)\qquad \partial_{\nu}\{g^{\mu\nu}(-g)^{\frac{1}{2}}\}=0$$

($\partial_\nu = \partial/\partial x^\nu$; Greek letters run 1, 2, 3, 4) in terms of the two metric tensors $g_{\mu\nu}$ and $\gamma_{\mu\nu}$ of Rosen's relativity theory [N. Rosen, Phys. Rev. (2) **57**, 147–150, 150–153, 154–155 (1940); MR **1**, 183]. The second tensor becomes the Minkowskian tensor

$$\gamma_{\mu\nu} = \mathrm{diag}\,\{-1, -1, -1, 1\} \tag{2}$$

in a certain class of coordinate systems.

In the paper under review the author discusses those transformations which preserve (1) and (2). He believes that (1) must have some physical significance because of the simplification it produces in Einstein's field equations.

Although he recognizes that (1) is not covariant, the author states "the Schwarzschild solution for the static field around a point mass at rest does not satisfy condition (1)". This appears to mean that (1) is not satisfied by the Schwarzschild metric if it is transformed from the non-isotropic form

$$(*)\quad ds^2 = (1-2m\rho^{-1})dT^2 - (1-2m\rho^{-1})^{-1}d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta d\phi^2)$$

to "rectangular" coordinates (T, x, y, z) defined by

$$(**)\quad x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta.$$

He rediscovers the result [C. Lanczos, Phys. Z. **23**, 537–539 (1922)] that (1) is satisfied by the Schwarzschild metric in the coordinates defined by (**) if ρ is replaced by $r = \rho - m$. He suggests that the Schwarzschild singularity may provide a cut-off radius which will eliminate some of the divergences of quantum field theory. *F. A. E. Pirani* (London).

Duan', I-Ši. Generalizations of the regular solutions of Einstein's equations of gravitation and of Maxwell's of electromagnetism for a point charge. Ž. Eksper. Teoret. Fiz. **27**, 756–758 (1954). (Russian)

A general solution of the gravitational and electromagnetic field equations for a point-charge at rest is found, such that the electric field and the metric tensor are everywhere free from singularities. *N. Rosen* (Haifa).

Bazzanella, Bruno. La teoria della materia e gli spinori. Boll. Un. Mat. Ital. (3) **10**, 59–60 (1955).

The author observes that the theory of Pfaffian forms connected with a quadratic form in $2n$ dimensions involves determinants of $2n+1$ dimensions which contain elements that may be considered as components of a non-symmetric second-order tensor. He suggests that this theory is related to the Kaluza theory of relativity and Einstein's generalized theory. The details of these relations are not given. *A. H. Taub* (Urbana, Ill.).

Raychaudhuri, Amalkumar. Relativistic cosmology. I. Phys. Rev. (2) **98**, 1123–1126 (1955).

Using a special coordinate system, the author derives a result equivalent to the following invariant one: The field equations for a pressure-free fluid (Greek indices range and sum over 1, 2, 3, 4 and a semi-colon denotes covariant differentiation)

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = -8\pi\rho v_\mu v_\nu$$

imply (*) $v^\mu{}_{;\nu}v^\nu = 0$ and (**) $R_{\mu\nu}v^\mu v^\nu = \Lambda - 4\pi\rho$. From the integrability conditions $v_{\rho;\mu\nu} - v_{\rho;\nu\mu} = R^\sigma{}_{\rho\mu\nu}v_\sigma$ and (*) and (**) one may deduce

$$\frac{d}{ds}(v^\rho{}_{;\rho}) + \tfrac{1}{3}(v^\rho{}_{;\rho})^2 = \Lambda - 4\pi\rho - \phi^2 + 2\omega^2$$

where

$$\phi^2 = \alpha_{\mu\nu}\alpha^{\mu\nu},\quad 2\omega^2 = \omega_{\mu\nu}\omega^{\mu\nu}$$

and

$$\omega_{\mu\nu} = \tfrac{1}{2}(v_{\mu;\nu} - v_{\nu;\mu})$$

$$\alpha_{\mu\nu} = \tfrac{1}{2}(v_{\mu;\nu} + v_{\nu;\mu}) - \tfrac{1}{3}(g_{\mu\nu} - v_\mu v_\nu)v^\tau{}_{;\tau}.$$

Here $\omega_{\mu\nu}$ is identified physically with spin, while ϕ^2 vanishes at a point if and only if the world-expansion (or contraction) about that point is isotropic. He deduces some other results about world-fluids with spin and anisotropic expansion, and concludes that the introduction of anisotropy without spin would not resolve the time-scale difficulty in an acceptable way. [However, the time-scale difficulty has resolved itself since the revision of the Cepheid observations and the distance scale. Cf. Sandage, Publ. Astr. Soc. Pacific **67**, 115–116 (1955).] *F. A. E. Pirani* (London).

Halatnikov, I. M. Some questions of relativistic hydrodynamics. Ž. Eksper. Teoret. Fiz. **27**, 529–541 (1954). (Russian)

A variational principle is set up, and from it the equations of relativistic hydrodynamics are derived. The relativistic analog of potential flow is discussed, and the generalized Bernoulli equations are obtained for this case. Some properties of shock waves are investigated, and sound waves of large amplitude are treated. The general one-dimensional isentropic flow problem is solved exactly. *N. Rosen.*

MECHANICS

Raher, W., und Selig, F. Die Verwendung der Motorsymbolik in der theoretischen Mechanik. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. II. **163**, 123–145 (1954).

Die Zusammensetzung infinitesimaler Bewegungen eines starren Körpers und die Reduktion der an ihm angreifenden Kräften sind bekanntlich liniengeometrisch identische Probleme. Die von Möbius, Plücker, Ball und Study gegebenen Entwickelungen fortsetzend hat von Mises [Z. Angew. Math. Mech. **4**, 155–181, 193–213 (1924)] seine Motorrechnung aufgebaut, woran die Verfasser anknüpfen. Der Geschwindigkeitszustand des Körpers ist durch den Geschwindigkeitsmotors

$$\mathfrak{G} = \left(\mathfrak{W}; \frac{d\mathfrak{R}}{dt} + \mathfrak{R}\times\mathfrak{W}\right)$$

gegeben, wobei $\mathfrak{W}$ die Winkelgeschwindigkeit, $\mathfrak{R}$ der Ortsvektor OO' vom raumfesten Punkt O zum körperfesten Punkt O' darstellt. Wenn man wie üblich $\mathfrak{W}$ mittels Quasikoordinaten als Zeitdifferential durch $d\overline{\Pi}/dt$ angibt so kann man formal $\mathfrak{G} = d\mathfrak{S}/dt$ setzen. Die Lagenkoordinaten werden als Funktionen der Zeit und eines Variationsparameters ϵ aufgefasst; der Motor $\partial\mathfrak{S}/\partial\epsilon = \mathfrak{G}^V$ ist die "Lagenvariation", und die Verf. beweisen die Vertauschungsrelation

$$\frac{\partial\mathfrak{G}}{\partial\epsilon} - \frac{\partial\mathfrak{G}^V}{\partial t} = \mathfrak{G}^V\times\mathfrak{G}.$$

Wenn der Kraftmotor $\mathfrak{K}$ des am Körper angreifenden Kraftsystems nur von der Lage abhängt und das Arbeitsdifferential vollständig ist, besteht die Potentialfunktion U und die Verf. zeigen $\mathfrak{K} = -\partial U/\partial\mathfrak{S}$. Es wird der Impulsmotor $\mathfrak{J}$ eingeführt und gezeigt $\mathfrak{J} = \mathbf{T}\mathfrak{G}$, wo $\mathbf{T}$ eine motorische Trägheitsdyade ist. Die kinetische Energie ist $\frac{1}{2}(\mathbf{T}\mathfrak{G})\mathfrak{G}$ und man hat $d\mathfrak{J}/dt = \mathfrak{K}$. Verf. geben noch mittels Motorrechnung

Formulierungen der Prinzipien der Mechanik und der Appelschen Gleichungen. *O. Bottema* (Delft).

Bilimović, A. Sur le centre de déviation. Srpska Akad. Nauka. Zb. Rad. 43, Mat. Inst. 4, 63–66 (1955). (Serbo-Croatian. French summary)

Let G be the vector of angular momentum of a gyrating rigid body, w the vector of angular velocity, J the moment of inertia about the axis of w. The vector $Jw-G=H$ is the component of G across w, and (since Hw^2 is homogeneous cubic in w), the vector $H/|w|$ will depend on the direction of w only. The components of Hw^2 are (using the familiar notation) $(B-A)q^2+(C-A)r^2$, etc. The author calls the terminal of vector $H/|w|$ the center of deviation.
A. W. Wundheiler (Chicago, Ill.).

Schlögl, Elmar. Über besondere Bewegungsformen des mathematischen Pendels. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. 1954, 81–97 (1955).

The author treats a simple pendulum on a string assumed to be without mass but perfectly inextensible and without any bending stiffness. For great amplitudes in an interval, as is already to be found in the first ed. of Appell's Traité de mécanique rationnelle [T. I, Gauthier-Villars, Paris, 1893, p. 422], this pendulum makes an overturn (Überschlag), i.e. it passes from the circular path to a parabolic projectile path. The author analyses in an elementary way the loss of kinetic energy on return of the pendulum to the circular path as well as the number of the possible overturns in its dependence on the initial velocity.
T. P. Anđelić.

Savin, G. N. On dynamic forces in a shaft lifting cable while lifting a load. Ukrain. Mat. Ž. 6, 126–139 (1954). (Russian)

This paper is a somewhat more detailed version of a previously reviewed paper [Dokl. Akad. Nauk SSSR (N.S.) 97, 991–994 (1954); MR 16, 533].
E. Leimanis.

De Caro, E. Espressione generale della velocità areolare nel moto relativo di due corpi di masse variabili. Atti Accad. Gioenia Catania (6) 9, 12–18 (1954).

Popoff, Kyrille. Construction de la famille de trajectoires d'un projectile d'après les données directes obtenues des tirs aux polygones. Bŭlgar. Akad. Nauk. Izv. Mat. Inst. 1, no. 2, 47–66 (1954). (Bulgarian. Russian and French summaries)

Let O be the center of the muzzle, Oz the descending vertical, Oy an axis in the direction of the muzzle velocity, α the angle of elevation, and v the instantaneous speed of the projectile. The author assumes the equations of motion to be

$$x=0,\quad \ddot{y}=-\dot{y}f(v),\quad \ddot{z}=g-\dot{z}f(v),$$

where $vf(v)$ is positive and analytic for positive v, and vanishes for $v=0$. He proposes to represent the family of plane trajectories in one of the two forms:

$$y=\sum y_i(t)\sin^i\alpha,\quad z=\cdots,$$

or

$$y=\sum y_i(t)\sin^{2i}\phi,\quad z=\cdots\quad (i=0,1,\cdots),$$

where $\phi=\frac{1}{4}\pi+\frac{1}{2}\alpha$. *A. W. Wundheiler* (Chicago, Ill.).

Hydrodynamics, Aerodynamics, Acoustics

Ericksen, J. L. On the uniqueness of ideal gas flows with given streamline patterns. Bull. Tech. Univ. Istanbul 6, 1–5 (1953). (Turkish summary)

The author considers steady, plane, rotational flows of a nonviscous, thermally non-conducting ideal gas. He shows that, in general, there can be at most three different flows of this type having the same streamline pattern. Exceptions to this occur only if the flows are of vortex or source-sink type. (Two flows are considered different if one cannot be reduced to the other by a Munk-Prim transformation.) This result supplements earlier work of Gilbarg [J. Math. Phys. 26, 137–142 (1947); MR 9, 252] and Prim [ibid. 28, 50–53 (1949); MR 10, 634] on the uniqueness of flows with a given streamline pattern. *J. B. Serrin.*

Yih, Chia-Shun. Maximum acceleration in two-dimensional steady flows of an ideal fluid. Quart. Appl. Math. 13, 202–203 (1955).

The author notes that, for steady two-dimensional irrotational flow of an incompressible fluid the magnitude of the acceleration is given by $|w'w''|$, where $w=w(z)$ is the complex velocity potential. He infers as a consequence that the maximum acceleration must occur at the boundary of a flow. *J. B. Serrin* (Minneapolis, Minn.).

Müller, W. Über den Einfluss der Vergrösserung des Kopfteiles eines in der Flüssigkeit bewegten Rumpfkörpers auf die Trägheitskoeffizienten und das instabile Drehmoment. Österreich. Ing.-Arch. 9, 1–11 (1955).

A prolate spheroid moving axially in inviscid liquid can be replaced by a linear system of sources and sinks. A suitable doublet placed at the forward end gives rise to a new tadpole form of stream surface whose after portion differs but little from the spheroid and whose nose differs but little from a sphere. By means of Legendre functions the author calculates, approximately, the inertia coefficients and moment when a body of this form is placed at incidence in a stream. Comparison of the results with those for a spheroid of equal length and volume show a reduction in the steering moment of 40 per cent in favour of the tadpole form. Some generalizations are indicated. *L. M. Milne-Thomson.*

Breslin, John P. Two-dimensional flow about half bodies between parallel walls. J. Appl. Mech. 22, 35–40 (1955).

The author considers two families of half bodies in two-dimensional flow, formed by the superposition of a source distribution and a uniform stream. One is obtained from a source lamina placed normal to an infinite free stream, the other is obtained by using the same distribution between parallel walls. The point of these particular source distributions is that the resulting half bodies have very blunt, almost flat, leading edges, a useful feature in the discussion of some cavitation phenomena. Formulas are given for the half bodies thus obtained, together with graphs for several values of the parameters. The influence of the walls on the body shape and pressure distribution is also studied.
J. B. Serrin (Minneapolis, Minn.).

Belocerkovskiĭ, S. M. Horseshoe-shaped vortex for unsteady motion. Prikl. Mat. Meh. 19, 159–164 (1955). (Russian)

The author first finds the downwash distribution due to a horseshoe vortex of fixed strength, and then deduces the corresponding distribution for a horseshoe in which the

strength of the bound vortices depends on time and free vortices parallel to these are shed at a constant rate. Finally, a system of bound vortices with strength varying harmonically with time is considered, and the results for this are applied to a wing oscillating harmonically with small amplitude. The author states that the results can be extended to cover unsteady motion of a slightly curved wing but it is not clear from the paper how this should be done. The results are left in general form and no numerical example is included. *M. Holt* (Cambridge, Mass.).

Dorfman, L. A. The inverse problem for a grid of profiles. Prikl. Mat. Meh. 18, 637–640 (1954). (Russian)

Dans la technique aéronautique courante, on a besoin de calculer la forme des profils des éléments d'une persienne, connaissant la répartition des vitesses le long du profil. L'A. résout rigoureusement ce problème, en utilisant la mèthode de Sedov, une fois admise l'approximation de Čaplygin pour relier la vitesse à la densité. *J. Kravtchenko* (Grenoble).

Hažaliya, G. Ya. On steady motion of a fluid in a tube deviating slightly from a cylinder. Dokl. Akad. Nauk SSSR (N.S.) 95, 465–468 (1954). (Russian)

Généralisant une formule de Lavrentiev, valable dans le cas du plan, l'A. donne le mouvement à potentiel et à symétrie axiale du liquide parfait, non pesant, dans un tube indéfini, peu différent d'une cylindre de révolution. Ou peut même construire un tel écoulement comportant une source et un puit sur l'axe. Comme application, l'A. détermine les mouvements par ondes dans un tube cylindrique élastique, en supposant grande la longueur d'onde par rapport au diamètre de l'écoulement. *J. Kravtchenko* (Grenoble).

Woods, L. C. Unsteady plane flow past curved obstacles with infinite wakes. Proc. Roy. Soc. London. Ser. A. 229, 152–180 (1955).

A theory of unsteady flow about obstacles behind which are wakes or cavities of infinite extent is developed for the case when the velocities and displacements of the unsteady perturbations about the mean steady motion are small. Unsteady Helmholtz flows (constant wake pressure) receive detailed attention, both for general non-uniform motion and for the special case of harmonic motions of long duration. A number of possible applications of the theory to aerodynamic problems are indicated, the most important being the flutter of a stalled aerofoil. The classical theory of unsteady aerofoil motion is shown to be a special case of the theory given in this paper. (Author's summary.)
D. Gilbarg (Stanford, Calif.).

Woods, L. C. Two-dimensional flow of a compressible fluid past given curved obstacles with infinite wakes. Proc. Roy. Soc. London. Ser. A. 227, 367–386 (1955).

This paper extends in several ways the classical Helmholtz-Kirchhoff theory of incompressible flow about obstacles behind which are constant-pressure wakes or cavities. In the first place, as in the author's earlier paper [Quart. J. Mech. Appl. Math. 7, 263–282 (1954); MR 16, 418], compressibility is taken into account by making use of a Kármán-Tsien approximation. The problem is then reduced to an integral equation, and an iterative scheme is developed for the numerical solution (here again, see the paper referred to above). Finally the author presents a modification of the Helmholtz-Kirchhoff model in order to account for non-zero cavitation numbers and to give a finite width to the wake at infinity. As an example, the flow is calculated past a circular cylinder for a number of points of separation and Mach numbers. When the flow is incompressible and the points of separation are the same as those found experimentally, the theoretical and experimental pressure distributions over the cylinder are in good agreement. The difficulty of numerical work is known to research workers in free-boundary theory, and therefore these calculations and the method used should be of interest to them.
J. B. Serrin (Minneapolis, Minn.).

Emersleben, O. Über eine doppelperiodische Parallelströmung zäher Flüssigkeiten. Z. Angew. Math. Mech. 35, 156–160 (1955).

The author considers generalized Poiseille flows arising by superposition from a particular doubly periodic solution of the Poisson equation, namely the Epstein Zeta function.
J. B. Serrin (Minneapolis, Minn.).

Dolidze, D. E. Uniqueness of solution of the fundamental boundary problem of a viscous incompressible fluid. Dokl. Akad. Nauk SSSR (N.S.) 96, 437–439 (1954). (Russian)

L'A. donne un aperçu de sa démonstration du théorème d'unicité relatif aux solutions régulières des équations de Navier, dans le cas des liquides visqueux: étant donné un domaine D, dont la frontière σ se compose d'un nombre fini de morceaux de surfaces lisses, il ne peut exister qu'une seule solution au plus, se réduisant pour $t=0$ à des valeurs données dans D et prenant sur σ pour $t>0$ les valeurs données. La démonstration de l'A. suppose la continuité des dérivées de la vitesse dans le domaine fermé $D+\sigma$. Le résultat, d'après l'A., peut être étendu aux domaines non bornés.
J. Kravtchenko (Grenoble).

Dolidze, D. E. Unsteady motion of a viscous fluid created by a rotating disc. Prikl. Mat. Meh. 18, 371–378 (1954). (Russian)

Une plaque plane, indéfinie, est animée d'un mouvement de rotation non uniforme (connu a priori) autour d'un axe perpendiculaire à son plan. La plaque est placée dans un liquide visqueux indéfini, soustrait à l'action des forces extérieures. A l'instant initial le liquide est au repos; les équations du problème ne sont pas linéarisées. L'A. parvient à déterminer le mouvement du liquide. A cet effet, il utilise une transformation de Kármán pour ramener la question à un système intégro-différentiel non linéaire; ce système est résolu par un processus convergent d'approximations successives. La solution se prête mal aux calcul numériques; l'A. se borné à l'examen des deux premiers termes des développements. *J. Kravtchenko* (Grenoble).

***Dolidze, D. E. Unsteady motion of a viscous fluid created by a rotating disc.** Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 10 pp. (mimeographed). $5.00.

Translation of the paper reviewed above.

Kiselev, A. A. On unsteady flow of a viscous fluid in the presence of external forces. Dokl. Akad. Nauk SSSR (N.S.) 100, 871–874 (1955). (Russian)

L'A. donne un exposé sommaire de ses travaux consacrés à la détermination des mouvements plans, non-linéaires et non-stationnaires des liquides visqueux. Soit:

$$Q=\Omega\times[0\leqq t\leqq T],$$

où Ω est un domaine plan borné du plan Ox_1x_2, limité par la courbe S; T un constante >0, fonctionnelle de Ω, des conditions initiales et des forces de masse. La question se ramène à la détermination dans Q d'une solution de:

$$(1)\qquad \frac{\partial\Delta\psi}{\partial t}-\nu\Delta^2\psi+\frac{\partial\psi}{\partial x_2}\frac{\partial\Delta\psi}{\partial x_1}-\frac{\partial\psi}{\partial x_1}\frac{\partial\Delta\psi}{\partial x_2}=f(x_1,x_2,t),$$

où f est une donnée, avec les conditions aux limites:

$$(2)\quad \psi|_{t=0}=\psi_0(x_1,x_2);\qquad (3)\quad \psi\Big|_S=\frac{\partial\psi}{\partial n}\Big|_S=0\quad\text{pour}\quad t\geqq 0,$$

où $\psi_0(x,y)$ est donnée. L'A. nomme solution généralisée de (1), (2), (3) une fonction Ψ, continue dans Q et telle que: 1) Ψ et $\partial\Psi/\partial t$ admettent des dérivées partielles en x_i $(i=1,2)$ de deux premiers ordres de carré sommable sur Q; 2) Ψ vérifie (2) et (3); 3) Ψ vérifie l'identité:

$$\int_Q\left\{\sum_i\frac{\partial^2\Psi}{\partial t\partial x_i}\frac{\partial\Phi}{\partial x_i}+\nu\Delta\Psi\Delta\Phi+\frac{\partial\Psi}{\partial x_2}\Delta\Psi\frac{\partial\Phi}{\partial x_1}-\frac{\partial\Psi}{\partial x_1}\Delta\Psi\frac{\partial\Phi}{\partial x_2}+f\Phi\right\}dQ=0,$$

pour toute fonction Φ deux fois continûment différentiable en x_i et satisfaisant à (3). L'A. montre d'abord qu'une telle fonction Ψ est unique. Moyennant quelques hypothèses de régularité, l'A. démontre l'existence de Ψ au moyen de la methode aux différences finies, telle qu'elle est exposée dans O. A. Ladyženskaya [Problèmes mixtes de type hyperbolique, Gostehizdat, Moscou, 1953]. Sous certaines conditions complémentaires, l'A. démontre que $\Psi\to 0$ pour $t\to\infty$ ceci uniformément dans $\bar\Omega$. Il reste à identifier Ψ et ψ. Ce mémoire complète heureusement les travaux classiques de Leray. *J. Kravtchenko* (Grenoble).

Kiselëv, A. A. On the solution of the equations describing the motion of an incompressible fluid. Uspehi Matem. Nauk (N.S.) 9, no. 4(62), 251–254 (1954). (Russian)

L'A. résume son mémoire analysé ci-dessus.

J. Kravtchenko (Grenoble).

Kiselev, A. A., and Ladyženskaya, O. A. On the solution of the linearized equations of a plane unsteady flow of a viscous incompressible fluid. Dokl. Akad. Nauk SSSR (N.S.) 95, 1161–1164 (1954). (Russian)

Les A. déterminent les écoulements plans, linéarisés (mais non stationnaires) d'un liquide visqueux dans le domaine Ω pour $t\geqq 0$; Ω est un domain borné simplement connexe ou le complémentaire d'un tel domaine (dans le second cas on admettra qui le mouvement est uniforme à l'infini). Tout revient, comme on sait, à construire dans Ω une solution régulière de

$$(1)\qquad \frac{\partial\Delta\psi}{\partial t}-\Delta^2\psi=0,$$

satisfaisant à

$$(2)\qquad \psi|_{t=0}=\psi_0(X)$$

où $X\in\Omega$ et où $\psi_0(X)$ est une donnée, et à

$$(3)\qquad \psi\Big|_S=\frac{\partial\psi}{\partial n}\Big|_S=0,\quad\text{pour}\quad t\geqq 0.$$

où S est la frontière de Ω.

Les A. cherchent la solution sous forme du développement de Fourier:

$$(4)\qquad \psi=\sum_{k=1}^{\infty}c_ke^{-\lambda_kt}\varphi_k(X),$$

où les c_k sont des constantes, λ_k et φ_k étant les valeurs et les fonctions propres de $\Delta^2\varphi_k+\lambda_k\varphi_k=0$ avec les conditions frontières (3). Après avoir établi les propriétés habituelles des développements de type (4), les A. justifient d'abord l'unicité, puis l'existence (ceci moyennant quelques hypothèses de régularité concernant les données) d'une solution généralisée sous la forme (4), du problème (1), (2), (3). Si la norme (au sens de Hilbert) de ψ_0 sur Ω est bornée, on a, de plus, la propriété: $\psi\to 0$ pour $t\to\infty$. *J. Kravtchenko.*

Petuhov, I. V. Inertialess laminar flows of a viscous gas in plane channels. Prikl. Mat. Meh. 18, 385–398 (1954). (Russian)

L'A. forme d'abord les équations de l'écoulement laminaire permanent d'un gas visqueux en coordonnées curvilignes orthogonales quelconques; il est tenu compte des échanges thermiques dans la masse du fluide, mais les pertes par rayonnement sont négligées. Les résultats obtenus sont particularisés en supposant les mouvements plans, la famille des lignes de courant étant prise pour l'une des familles des courbes de référence. On peut alors énoncer le problème aux limites pour déterminer l'écoulement d'un gaz dans un canal limité par deux parois. L'A. particularise encore et cherche les mouvements pour lequels les forces d'inertie s'annulent identiquement. De tels régimes se partagent en trois classes; pour chacune d'elles le problème posé peut être résolu et les solutions obtenues sont calculables numériquement. Les résultats sont tabulés. *J. Kravtchenko.*

Cukker, M. S. A laminar incompressible jet streaming from a radial diffusor along a wall. Prikl. Mat. Meh. 18, 757–761 (1954). (Russian)

L'A. forme le système aux dérivées partielles que vérifient les vitesses dans un jet liquide laminaire, tordu sur lui-même, s'échappant d'un orifice circulaire; il est tenu compte de l'effet de la paroi de l'appareil. En particulier, l'A. obtient les relations intégrales que doit satisfaire le jet pour tenir compte de la couche limite. Le système différentiel obtenu est résolu en première approximation. Il faut souligner l'intérêt physique de l'étude qui se rapporte au fonctionnement, en régime permanent, d'un diffuseur radial.

J. Kravtchenko (Grenoble).

Slezkin, N. A. Remark on the notes of Yu. V. Rumer, "The problem of a submerged jet", and of L. G. Loĭcyanskiĭ, "Propagation of a whirling jet into an infinite space filled with the same fluid". Prikl. Mat. Meh. 18, 764 (1954). (Russian)

Dans leur livre bien connu "Mécanique des milieux continus" [Moscou, 1944, §19; 2ème éd., 1953, §23, pp. 108–110; MR 16, 412], Landau et Lifšic ont publié la solution explicite du problème de la veine liquide noyée dans un liquide visqueux emplissant l'espace. Diverses publications ont attribué ce resultat aux auteurs précités. L'A. signale que le calcul en cause figure déja explicitement dans une note publiée par lui [Moskov. Gos. Univ. Uč. Zap. 2, 89–90 (1934)]. *J. Kravtchenko* (Grenoble).

Položil, G. N. The method of movement of boundary points and majorant regions in the theory of filtration. Ukrain. Mat. Ž. 5, 380–400 (1953). (Russian)

M. Lavrent'ev et M. Keldyš ont fait un usage heureux des méthodes variationnelles dans divers problèmes de représentation conforme. L'A. applique ces raisonnements à l'étude des écoulements plans des liquides pesants en milieux poreux. Considérons un domaine D, occupé par le liquide en

mouvement dont l'image conforme dans le plan du potentiel complexe soit un rectangle, de côtés paralleles aux axes; il s'agit d'étudier l'éffet sur le régime des variations, connues a priori, des frontiéres de D ou de ses images dans les différents plans auxiliaires. Les résultats semblent très intéressants, en particulier, lorsque la frontière de D contient une ligne de suintement. *J. Kravtchenko* (Grenoble).

Mirzadžanzade, A. H. Unsteady motion of a viscous-plastic fluid in a cylindrical tube of circular cross-section. Dokl. Akad. Nauk SSSR (N.S.) **95**, 947–950 (1954). (Russian)

L'A. construit une solution non stationnaire, correspondant à une distribution des vitesses connues à l'instant initial, des équations de l'écoulement d'un liquides, à la fois plastique et visqueux, dans un tube à section circulaire. Le régime est supposé de révolution autour de l'axe du tube. Des approximations convenables permettent de tenir compte de la présence d'un couche purement plastique. A noter que dans les travaux antérieurs, la vitesse le long de l'axe pouvait être orientée dans le sens contraire du mouvement général. La solution de l'A. échappe a cet inconvénient.

J. Kravtchenko (Grenoble).

Mirzadžanzade, A. H. Immersion of a thin cylindrical tube in a viscous-plastic fluid. Dokl. Akad. Nauk SSSR (N.S.) **99**, 511–514 (1954). (Russian)

A l'instant $t=0$, on commence à immerger un tube fini de révolution (dont l'axe sera maintenu vertical et la vitesse de translation sera constante) dans un milieu visco-plastique indéfini, limité par un plan horizontal. Moyennant quelques approximations, l'A. construit, sous forme de développements en séries de fonctions de Bessel, le champs des vitesses du milieu et calcule les efforts subis par le tube.

J. Kravtchenko (Grenoble).

Kasimov, A. F., and Mirzadžanzade, A. H. Different forms of the equations of motion of viscous-plastic fluids and the law of hydrodynamic similarity. Prikl. Mat. Meh. **19**, 348–352 (1955). (Russian)

The authors consider ideal materials for which the stress deviator is proportional to the deviator of the rate-of-deformation tensor. The factor of proportionality is assumed to be of the form $\eta+\tau_0/h$, where η and τ_0 are positive constants and h^2 is the trace of the square of the rate-of-deformation matrix. The authors give various forms of the corresponding equations of motion. A solution for flow in a circular tube is derived for incompressible materials. The paper concludes with a discussion of dynamical-similarity parameters.

J. L. Ericksen (Washington, D. C.).

Tyabin, N. V. Unsteady flow of a viscous-plastic medium in a circular tube. Dokl. Akad. Nauk SSSR (N.S.) **95**, 473–475 (1954). (Russian)

L'A. construit une solution des équations de la viscosité plastique (qui lui sont dues) dans le cas particulier de l'écoulement non permanent dans un tube cylindrique. On suppose connue a priori la pression, alors que c'est une inconnue du problème. L'A. se borne à supposer les vitesses finies, sans se préoccuper de leurs signes. D'après l'A. ses calculs sont en bon accord avec les mesures de contrôle, effectuées avec l'appareil de Vinogradov et Konstantinov.

J. Kravtchenko (Grenoble).

Wadhwa, Y. D. Boundary layer for a parabolic cylinder. Z. Angew. Math. Mech. **35**, 67–69 (1955).

In this short note the boundary layer over a parabolic cylinder is treated by a method due to Seth [Phil. Mag. (7) **27**, 212–220 (1939); Proc. Internat. Congress Math., Cambridge, Mass., 1950, vol. I, Amer. Math. Soc., Providence, R. I., 1952, pp. 636–637]. The idea is that appropriate body forces are introduced in the Navier-Stokes equations so that these equations become exactly integrable; if these forces can be made to vanish ultimately the solution will be physically acceptable. It is found that the thickness of the boundary layer over a parabolic cylinder is of the order of $R^{-k/(2+\lambda)}$, where $0<k<1$, $\lambda>0$ so that $k/(2+\lambda)<\frac{1}{2}$. The constants k and λ depend respectively on the vorticity allowable outside the boundary layer and the value of the Reynolds number, R. *R. C. DiPrima.*

Jain, M. K. Boundary layer effects in non-Newtonian fluids. Z. Angew. Math. Mech. **35**, 12–16 (1955). (English, French and Russian summaries)

Using second-order terms in the stress-strain velocity tensor relations the hydrodynamical equations governing the flow of non-Newtonian fluids have been set up and solved by employing a "synthetic method" due to B. R. Seth (see the preceding review). The uniform motion of a sphere and a circular cylinder are discussed in detail. (Taken from the author's summary.) *R. C. DiPrima.*

Šestopalov, V. P. Particular solution of a problem for the diffusion boundary layer in a diffusor. Prikl. Mat. Meh. **18**, 753–756 (1954). (Russian)

L'A. étudie le problème de la couche limite plane le long de la paroi rectiligne d'une diffuseur; le liquide est constitué par un mélange et il s'agit de tenir compte de la diffusion. La complexité du système d'équations aux dérivées partielles qui gouvernent le phénomène est telle qu'on est conduit à simplifier la question, en utilisant le fait que l'épaisseur de la couche limite de diffusion est très petite. Il est remarquable que l'A. a réussi à former une solution particulière exacte du problème, exprimée au moyen des séries hypergéometriques. Le calcul de l'épaisseur de la couche limite conduit aux nombres dont l'ordre de grandeur est en accord avec les vues actuelles. *J. Kravtchenko.*

Di Prima, Richard C. Application of the Galerkin method to problems in hydrodynamic stability. Quart. Appl. Math. **13**, 55–62 (1955).

The author successfully applies the Galerkin method to the calculation of the stability problem of the Taylor-Görtler type. For the stability of the motion between concentric cylinders rotating in opposite directions, it is pointed out, on theoretical grounds, that a certain parameter S should depend on a certain geometrical ratio related to the flow. This is borne out by an analysis of the experimental data. Theoretical calculations also check the experimental values. *C. C. Lin* (Cambridge, Mass.).

Tsuji, Hiroshi. The transformation equations between one- and n-dimensional spectra in the n-dimensional isotropic vector or scalar fluctuation field. J. Phys. Soc. Japan **10**, 278–285 (1955).

L'auteur se place dans un milieu turbulent isotrope à n dimensions et établit les relations directes et réciproques entre 1) le tenseur $R_{ij}(\mathbf{r})=\overline{u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})}$ et les diverses fonctions spectrales qui lui sont associeés; 2) le scalaire

$M(\mathbf{r})=\overline{\theta(\mathbf{x})\theta(\mathbf{x}+\mathbf{r})}$ et les diverses fonctions spectrales qui lui sont associeés. *J. Bass* (Chaville).

Reid, W. H. On the stretching of material lines and surfaces in isotropic turbulence with zero fourth cumulants. With an appendix by G. K. Batchelor. Proc. Cambridge Philos. Soc. **51**, 350–362 (1955).

Cette étude est le prolongement de travaux de G. K. Batchelor [Proc. Roy. Soc. London. Ser. A. **213**, 349–366 (1952); MR **14**, 698] et de I. Proudman et W. H. Reid [Philos. Trans. Roy. Soc. London. Ser. A. **247**, 163–189 (1954); MR **16**, 299]. L'auteur considère un milieu turbulent isotrope donné et il étudie l'effet de la turbulence sur un champ scalaire $\vartheta(\mathbf{x})$ ou un champ vectoriel solénoidal $\mathbf{F}(\mathbf{x})$. Tout repose sur l'hypothèse que les moyennes de produits de 4 facteurs formés à partir de ϑ ou de $\mathbf{F}$ et de la vitesse $u_i(\mathbf{x})$ sont reliées aux moyennes de produits de 2 facteurs comme dans le cas d'une loi normale.

1. Champs scalaires. L'auteur établit des équations d'évolution linéaires pour

$$\theta(\mathbf{r})=\overline{\vartheta(\mathbf{x})\vartheta(\mathbf{x}+\mathbf{r})} \quad \text{et} \quad \theta_i(\mathbf{r},\mathbf{r}')=\overline{u_i(\mathbf{x})\vartheta(\mathbf{x}+\mathbf{r})\vartheta(\mathbf{x}+\mathbf{r}')}.$$

Il les transforme en tenant compte de l'isotropie, puis passe en termes spectraux. Il montre en particulier que, si $\overline{G^2}$ est le carré scalaire moyen de grad ϑ,

$$\frac{d^2\overline{\mathrm{G}^2}}{dt^2}=\tfrac{2}{3}\overline{\omega^2}\,\overline{\mathrm{G}^2}$$

où $\overline{\omega^2}$ est le carré moyen du tourbillon. Discussion avec diverses hypotheses sur $\overline{\omega^2}$.

2. Champs de vecteur. Aux tenseurs

$$\overline{F_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})} \quad \text{et} \quad \overline{u_i(\mathbf{x})F_j(\mathbf{x}+\mathbf{r})F_k(\mathbf{x}+\mathbf{r}')}$$

on peut associer des tenseurs spectraux qui, grâce à l'isotropie, sont définis par deux scalaires $H(k)$ et $\Xi(k)$. L'auteur montre que l'évolution de $H(k)$ est régie par une équation fonctionelle linéaire, et celle de $\Xi(k)$ par une équation qui contient des termes quadratiques en $H(k)$. Ces équations sont appliquées à l'étude de $\overline{\mathbf{F}^2}$ et $\overline{\mathbf{F}\cdot\omega}$, pour les grandes valeurs du temps et avec diverses hypothèses sur $\overline{\omega^2}$, et à une courte étude de la déformation d'un élément de volume sous l'action de la diffusion turbulente.

Dans l'appendice, G. K. Batchelor apporte une correction à une des ses publications (référence ci dessus). Lorsqu'un élément de volume matériel suit le mouvement turbulent, une de ses dimensions finit par s'allonger, et les deux autres deviennent très petites, mais, contrairement à ce qui avait été supposé, elles ne sont pas forcément du même ordre de grandeur. La section peut devenir plate. *J. Bass.*

Richter, H., und Müller, W. Zur Tschaplyginschen Hodographenmethode bei Unterschallströmungen mit Zirkulation. Z. Angew. Math. Mech. **35**, 1–11 (1955). (English, French and Russian summaries)

L'article est consacré à l'étude des écoulements subsoniques d'un fluide compressible satisfaisant à la loi d'état de Chaplygin, la pression étant une fonction linéaire du volume spécifique. Après avoir rappelé les formules bien connues permettant de ramener le problème à la résolution d'une équation de Laplace, les auteurs étudient la possibilité d'une correspondance entre un écoulement de ce fluide compressible et un écoulement d'un fluide incompressible. Leur analyse inclut le cas de l'écoulement avec circulation autour d'un profil; des formules, généralisant les formules classiques dans le cas d'un fluide incompressible, sont données pour le calcul des efforts.

Ce sujet a fait déjà l'objet de nombreuses études; outre les références signalées dans la bibliographie de l'article, indiquons une note de P. Germain [C. R. Acad. Sci. Paris **223**, 532–534 (1946); MR **8**, 237] et un article de J. Leray [J. Math. Pures Appl. (9) **28**, 181–191 (1949); MR **11**, 475]. Dans ces deux publications, on trouve des considérations très voisines de celles développées par les auteurs. *P. Germain* (Paris).

Manwell, A. R. A family of plane compressible flows past a certain semi-infinite body. J. Math. Phys. **34**, 113–118 (1955).

L'auteur étudie une famille de solutions de le forme $f(q)\sin\theta$ pour l'équation vérifiée par la fonction de courant dans le plan de l'hodographe. On peut remarquer que la fonction $f(q)$ représente (à une constante additive près) la pression. Une ligne de courant est constituée par deux demi-droites et un arc de cycloïde. La fonction de courant est univalente dans le plan de l'hodographe; il n'en n'est pas de même dans le plan physique. La bibliographie est sommaire. *H. Cabannes* (Marseille).

Isay, Wolfgang-Hermann. Zur Behandlung der kompressiblen Unterschallströmung durch axiale und radiale Schaufelgitter. Z. Angew. Math. Mech. **35**, 34–44 (1955). (English, French and Russian summaries)

The Janzen-Rayleigh procedure is used here to solve approximately for the compressible flow in a cascade of thin blades. For the velocity (u, v) in an axial flow cascade, the equation $(\partial u/\partial x)+(\partial v/\partial x)=f(u, v)$, where

$$f(u,v)=c^{-2}\{(\partial u/\partial x)u^2+[(\partial u/\partial y)+(\partial v/\partial x)]uv+(\partial v/\partial y)v^2\}$$

and c is the local speed of sound, is solved by successive approximations (u_n, v_n) using

$$(\partial u_n/\partial x)+(\partial v_n/\partial y)=f(u_{n-1}, v_{n-1}) \quad (n=1, 2, \cdots).$$

The approximations are decomposed as $u_n=U_0+u_{n1}+u_{n2}$, $v_n=V_0+v_{n1}+v_{n2}$, where (U_0, V_0) is constant, where (u_{n1}, v_{n1}) is the velocity due to a vortex distribution $\gamma_n(x)$ along the blades with $\gamma_n(x)$ to be found by solving a certain integral equation, and where (u_{n2}, v_{n2}) is the velocity due to a potential of the form

$$\Phi_n(x, y)=\sum e^{\pm mpx}(a_{pq}\cos mqy+b_{pq}\sin mqy),$$

$m=2\pi/L$, $L=$blade pitch, with a_{pq} and b_{pq} determined from a similar expansion of $f(u_{n-1}, v_{n-1})$. The starting value (u_0, v_0) is taken as the velocity due to an incompressible flow through the same cascade with $\Phi_0(x, y)=0$. The same type of approximation is also developed for radial-flow cascades. The author states, however, that he has not been able to prove the convergence of the sequence (u_n, v_n). Reference is made to two of the author's earlier papers on incompressible flows through cascades [Z. Angew. Math. Mech. **33**, 397–409 (1953); Ing.-Arch. **22**, 203–210 (1954); MR **16**, 875]. *M. Marden* (Milwaukee, Wis.).

Iacob, Caius. Sur le mouvement parallèle au sol d'une plaque plane dans un courant fluide variable avec la hauteur. Acad. Repub. Pop. Romîne. Stud. Cerc. Mat. **5**, 333–349 (1954). (Romanian. Russian and French summaries)

A vertical plate is moving horizontally, with time-variable velocity, in an inviscid liquid bounded below by a horizontal bottom. The motion is two-dimensional and the liquid is endowed with constant vorticity. Reduction to a Dirichlet

problem enables the author to express the solution in terms of elliptic integrals. The cases of zero and non-zero circulations are investigated, and the force and moment are calculated. *L. M. Milne-Thomson* (Greenwich).

Iacob, Caius. Sur une généralisation de la règle de Joukowsky pour la détermination de la circulation. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 6, 221–227 (1954). (Romanian. Russian and French summaries)

In incompressible inviscid flow about a profile C presenting a cusp P, the circulation is k_0 and the spread at infinity is V. The author finds that in subsonic compressible flow about the same profile with the same velocity at infinity the circulation is

$$k_0(1-\tfrac{1}{4}M_0^2)+\tfrac{1}{2}\pi A^2M_0^2,$$

correct to M_0^2. Here $M_0=V/c_0$, c_0 being the speed of sound at the forward stagnation point, and A is the normal derivative, at the point corresponding to P on the circle into which C transforms, of the imaginary part of a certain function holomorphic outside C. *L. M. Milne-Thomson.*

Iacob, Caius. Recherches sur la théorie des mouvements coniques supersoniques. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 6, 603–622 (1954). (Romanian. Russian and French summaries)

The author has rederived results on the linearized conical flow about a thin delta wing with subsonic leading edges obtained by P. Germain [O.N.E.R.A. Publ. no. **34** (1949); MR **12**, 452]. After the usual transformation

$$\zeta=\xi+i\eta=\beta(x+iy)/\{1+[1-\beta^2(x^2+y^2)]^{1/2}\}$$

the velocity component normal to the wing $v(\xi,\eta)=\Re V(\zeta)$ for some analytic function $V(\zeta)$. For

$$\xi_1<\xi<\xi_2,\quad v(\xi,\pm 0)=\pm m(\xi)+n(\xi);$$

here $m(\xi)$ and $n(\xi)$ are determined by the shape of the wing; $v=0$ on $|\zeta|=1$; v behaves like $|\zeta-\xi_k|^{-1/2}$ near the leading edges $\zeta=\xi_k$, $k=1, 2$; and $dV/d\zeta=0$ for $\zeta=\pm 1$. Let $V(\zeta)=\sum_1^3 (v_j+iv_j')=\sum_1^3 V_j(\zeta)$ with the following boundary conditions. On $\xi_1<\xi<\xi_2$,

$$v_1(\xi,\pm 0)=\pm m(\xi);\quad v_2(\xi,\pm 0)=n(\xi)-\lambda$$

for an appropriately chosen real constant λ; $v_3(\xi,\pm 0)=\lambda$. On $-1<\xi<\xi_1$ and $\xi_2<\xi<1$, $v_1(\xi,0)=v_2'(\xi,0)=0$. On $|\zeta|=1$ all $v_j=0$. Then all of V_j can be expressed as definite integrals. The perturbation velocity component $w=\Re W(\zeta)$ parallel to the undisturbed flow, required for pressure calculations, can be obtained from $dW/d\zeta=-2i\beta^{-1}\zeta(\zeta^2-1)^{-1}dV/d\zeta$. Alternatively, the part due to angle of attack,

$$w_2+iw_2'=\Re W_2(\zeta)$$

defined by $dW_2/d\zeta=-2i\beta^{-1}\zeta(\zeta^2-1)^{-1}(dV_2/d\xi+dV_3/d\zeta)$ can be obtained as the solution of the mixed boundary-value problem

$$w_2'(\xi,0)=-2i\beta^{-1}\int_0^\xi \xi(\xi^2-1)^{-1}(dn/d\xi)d\xi+\text{const.},$$

for $\xi_1<\xi<\xi_2$, and $w_2=0$ on $|\zeta|=1$, of a type discussed by H. Villat [Acta Math. **40**, 101–178 (1915)]. The author also applies a suggestion of Germain's to compute the flow over a thin conical fuselage and flat plate delta wing combination that is symmetrical with respect to the plane of the wing. Cases of fuselages of circular and elliptical cross sections are worked out in detail. *J. H. Giese.*

Giese, J. H., and Cohn, H. Canonical equations for non-linearized steady irrotational conical flow. Quart. Appl. Math. 12, 351–360 (1955).

Ce travail est consacré à l'étude des écoulements coniques irrotationnels d'un fluide compressible non visqueux. Par définition, dans de tels écoulements, il existe un point 0 tel que, le long de chaque demi droite d'origine 0, le vecteur vitesse reste équipollent à lui même. L'image d'un tel écoulement dans l'espace de l'hodographe s'effectue sur une surface (Σ) et il est aisé d'obtenir l'équation aux dérivées partielles du second ordre quasi-linéaire et homogène dont (Σ) est une surface intégrale. Utilisant le paramétrage le plus général pour (Σ), les auteurs forment les équations aux dérivées partielles auxquelles satisfont les coordonnées de (Σ). Si l'équation est du type elliptique, on choisit un paramétrage isotherme; si elle est du type hyperbolique, on prend comme variables les coordonnées caractéristiques; il est alors facile d'obtenir les transformation les plus générales conservant le caractère canonique des équations ainsi obtenues. En particulier, dans le cas où l'équation est du type elliptique, on peut prendre comme domaine fondamental dans le plan des paramètres une couronne circulaire, l'une des circonférences limites représentant le cone obstacle, l'autre le cone de choc. Le problème aux limites qu'il s'agit de résoudre est naturellement extrêmement complexe.

Les auteurs appliquent ces considérations au cas connu de l'écoulement supersonique autour d'un cone de révolution. Quelques remarques, notamment sur la possibilité d'une solution numérique du problème, terminent cet article. *P. Germain* (Paris).

Fettis, Henry E. Tables of lift and moment coefficients for an oscillating wing-aileron combination in two-dimensional subsonic flow. Wright Air Development Center, Wright-Patterson Air Force Base, Ohio, AF Tech. Rep. **6688**, Supplement 1, v+94 pp. (1954).

Tamada, Kô, and Shibaoka, Yoshio. On supersonic flow past a finite wedge at the Crocco Mach number. J. Aero. Sci. 22, 261–263, 269 (1955).

Lorsque l'onde de choc de tête est attachée à la pointe avant d'un profil en coin et que l'écoulement en aval de cette onde et au voisinage du profil est subsonique, la fonction de courant ψ, exprimée dans le plan de l'hodographe, présente en général une singularité au point image du nez du profil. Il y a exception pour une valeur particulière du nombre de Mach amont (dépendant de l'angle du coin); ce nombre de Mach est le nombre de Mach de Crocco.

Les auteurs étudient précisément ce cas. Dans le plan de l'hodographe, il s'agit de résoudre un problème de Tricomi généralisé, en ce sens que sur l'arc tracé dans le demi plan elliptique, ψ est nulle sur une partie de l'arc, tandis que le rapport de ses dérivées est connu sur la partie complémentaire (qui est un arc de la polaire de choc). La solution qui ne présente pas de singularité dans le domaine, excepté au point image de l'épaulement du coin, peut s'exprimer à l'aide d'une série de "premières solutions de Darboux" [P. Germain et R. Bader, O.N.E.R.A. Publ. no. **54** (1952); MR **14**, 654]. Les auteurs pour le calcul numérique se limitent aux 4 premiers termes et satisfont donc approximativement seulement la condition de choc. Le résultat que l'on déduit de cette étude théorique concorde bien avec les résultats expérimentaux trouvés.

M. Yoshihara a étudié le problème analogue pour tout nombre de Mach correspondant à une onde de choc attachée avec écoulement subsonique à l'aval [Proc. 2nd U. S. Nat.

Congr. Appl. Mech., Ann Arbor, 1954, Amer. Soc. Mech. Engrs., New York, 1955, pp. 643–649]. *P. Germain.*

Fraenkel, L. E., and Portnoy, H. Supersonic flow past slender bodies with discontinuous profile slope. Aero. Quart. 6, 114–124 (1955).

La théorie de Ward [Quart. J. Mech. Appl. Math. **2**, 75–97 (1949); MR **10**, 644] qui permet de calculer les efforts aérodynamiques autour d'un corps effilé, est étendue au cas d'un profil dont la surface présente des points anguleux. La question est traiteé en résolvant l'équation linearisée du potentiel des vitesses au moyen d'une intégrale de Stieltjes. L'abscisse parallèlement à l'écoulement uniforme amont, étant désigné par s, et l'aire de la section du corps perpendiculairement à l'écoulement amont étant désignée par S, les auteurs établissent que la variation de la trainée en fonction du nombre de Mach dépend uniquement des discontinuités de la dérivée dS/ds. Les efforts latéraux et les moments sont également calculés. *H. Cabannes.*

Eula, Antonio. Caratteristiche aerodinamiche di ali a freccia con bordo d'attacco subsonico e bordo d'uscita supersonico. IV. Aerotecnica **34** (1954), 299–303 (1955).

[For parts I–III see Aerotecnica **30**, 107–113, 175–182 (1950); **31**, 103–109 (1951); MR **13**, 298.] Let W be an octohedral wing with sub-(super-)sonic swept-back leading (trailing) edges and no dihedral angle. Suppose W is symmetrical with respect to both the median plane and the plane containing the leading and trailing edges. Suppose the wing tips are clipped off symmetrically by planes parallel to the median plane. Computations performed for the author at zero angle of attack show a reduction of drag due to thickness, relative to the unclipped case, for certain ranges of leading and trailing edge sweep, location of maximum section thickness, and extent of truncation.
J. H. Giese (Havre de Grace, Md.).

De Schwarz, M. J. Appendice. Aerotecnica **34** (1954), 303–306 (1955).

Summary of the computations for the paper reviewed above. *J. Giese* (Havre de Grace, Md.).

Murgulescu, Elena. Le mouvement supersonique autour d'une aile Δ munie d'un fuselage conique. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **6**, 741–753 (1954). (Romanian. Russian and French summaries)

Il s'agit de l'écoulement supersonique d'un fluide parfait autour d'un obstacle conique constitué par une aile delta de faible ouverture et par un fuselage qui est un cone de révolution. Le problème est traité comme une application directe de la théorie générale des écoulements supersoniques linéarisés autour de cones élancés, développée dans le travail du rapporteur [Thèse, Paris, 1948; O.N.E.R.A. Publ. no. **34** (1949); MR **12**, 452]. On trouvera dans l'article les formules complètes permettant le calcul des vitesses et de la pression en fonction de l'incidence. [Il y a lieu de noter que l'auteur ne semble pas avoir eu la possibilité de connaitre les travaux parus, sur des questions extrêmement voisines, dans la littérature anglaise ou américaine.] *P. Germain.*

Runyan, Harry L., Woolston, Donald S., and Rainey, A. Gerald. Theoretical and experimental investigation of the effect of tunnel walls on the forces on an oscillating airfoil in two-dimensional subsonic compressible flow. NACA Tech. Note no. **3416**, 41 pp. (1955).

The analytical portion of this report has been published previously [D. Woolston and H. Runyan, J. Aero. Sci. **22**, 41–50 (1955); MR **16**, 537], but experimental results are also described in the present version. Reasonable agreement between theory and experiment is found.
J. W. Miles.

Watkins, Charles E., and Berman, Julian H. On the kernel function of the integral equation relating lift and downwash distributions of oscillating wings in supersonic flow. NACA Tech. Note no. **3438**, 43 pp. (1955).

The singularities of the kernel function are separated out in order to permit numerical calculation of the remainder, thereby extending a similar development for the subsonic problem [C. Watkins, H. Runyan and D. Woolston, NACA Tech. Note no. **3131** (1954); MR **15**, 474]. The reviewer notes that the analysis might have been simplified by a Laplace transformation with respect to the frequency parameter, as in the subsonic problem [G. N. Lance, J. Aero. Sci. **21**, 635–636 (1954); MR **16**, 87].
J. W. Miles (Los Angeles, Calif.).

Kiveliovitch, M. Quelques considérations sur le vent géostrophique. J. Sci. Météorol. **4**, 61–66 (1952). (English and Spanish summaries)

The author takes the three components of the geostrophic wind on a flat earth and these are used to obtain general expressions for the pressure and density. The case of a barotropic fluid is then considered and expressions are also obtained for the vorticity components in terms of spatial and time coordinates. *M. H. Rogers* (Urbana, Ill.).

Slezkin, N. A., and Šustov, S. N. On the stability of motion of a suspended particle in a laminar flow. Dokl. Akad. Nauk SSSR (N.S.) **96**, 933–936 (1954). (Russian)

Les A. étudient la stabilité du mouvement d'une particule sphérique pesante, animée d'un mouvement de rotation sur elle même, au sein d'un écoulement laminaire d'un liquide visqueux, de même densité que la particule. Dans le cas particulier des écoulements à distribution parabolique des vitesses, les A. forment les équations approchées du mouvement du corpuscule et en étudient la stabilité au moyen de la méthode de Liapounov. *J. Kravtchenko* (Grenoble).

★Slezkin, N. A., and Shustov, S. N. On the stability of particles suspended in a laminar flow. Morris D. Friedman, Two Pine Street, West Concord, Mass., 1954. 4 pp. (mimeographed). $3.00.

Translation of the paper reviewed above.

Trilling, Leon. On thermally induced sound fields. J. Acoust. Soc. Amer. **27**, 425–431 (1955).

Un fluide compressible, visqueux et conducteur est au repos et en contact avec une surface fixe. La surface étant chauffée, le gaz reçoit une quantité de chaleur donnée; l'auteur étudie le mouvement qui prend ainsi naissance. Les équations sont linéarisées et les mouvements supposées voisins du repos. Le cas des mouvements rectilignes est traité de façon explicite à l'aide de la transformation de Laplace. L'auteur étudie ensuite un problème plan; le fluide est situé dans le demi-plan $y>0$ et la surface chauffante est le demi-axe $x=0$, $y>0$. Ces divers problèmes présentent un intérêt incontestable pour l'étude des mouvements à grande vitesse, dans lesquels les phénomènes calorifiques sont importants; cet article traite le cas des mouvements lents, mais constitue un point de départ intéressant.
H. Cabannes.

Johansen, A., Olsen, H., and Wergeland, H. The force on obstacles in a sound field. Norske Vid. Selsk. Forh., Trondheim **28**, 16–20 (1955).

The ith component of force on an obstacle in a sound field is expressed as the surface integral of T_{in}, where n denotes the surface normal, T_{ik} is the tensor $-\phi_k(\partial L/\partial\phi_i)+\delta_{ik}\phi$, ϕ is the velocity potential, and L is the field Lagrangean. The result is applied to a sphere in the field of an infinite plane wave. *J. W. Miles* (Los Angeles, Calif.).

Awatani, Jobu. Studies on acoustic radiation pressure. I. General considerations. J. Acoust. Soc. Amer. **27**, 278–281 (1955).

In this paper, the radiation pressure is evaluated from the time average of the excess pressure that appears on the material surface under action of the waves. This radiation pressure is completely specified by the solution of the usual wave equation for any fluid. Irrotational motion is assumed. An expression for the radiation force in any direction is derived. The tensor property of the radiation pressure is then discussed. Finally, a general expression for a rigid object is obtained. *M. J. O. Strutt* (Zurich).

Awatani, Jobu. Studies on acoustic radiation pressure. II. Radiation pressure on a circular disk. J. Acoust. Soc. Amer. **27**, 282–286 (1955).

The expressions of the paper reviewed above are applied to a circular disc. Introducing spheroidal coordinates, appropriate solutions of the wave equation are given. From these the radiation force due to a plane progressive wave and also to a plane standing wave is calculated. Special cases of a small and of a large disk are discussed and curves for the radiation pressure are given in both cases.

M. J. O. Strutt (Zurich).

Martinek, Johann, and Yeh, Gordon C. K. Sound scattering and transmission by thin elastic rectangular plates. Quart. J. Mech. Appl. Math. **8**, 179–190 (1955).

The transmission coefficient of a thin, elastic, rectangular plate in an infinite plane baffle separating two fluid media of different properties is calculated. The motion of the plate is expressed in terms of its natural modes, while the field problem is treated on a quasi-one-dimensional approximation; the justification and limitations of this approximation are, in the reviewer's opinion, discussed in a rather inadequate and superficial manner. *J. W. Miles.*

Rahmatulin, H. A. Solution of the problem of reflection of sound waves from a rigid plane having a deformable part. Prikl. Mat. Meh. **18**, 573–584 (1954). (Russian)

A plane sound wave of given profile strikes directly at time $t=0$ the plane $z=0$, of which the strip $-l<x<l$ is deformable and initially at rest; for $t>0$ the motion of the strip is given by $v_z=f(x, t)$. The solution is based on the principle that the velocity potential at the front of the reflected wave is the same as its value ϕ_1 for a completely rigid plane. The true velocity potential is found to be

$$\phi_1(t, z)-\frac{a}{\pi}\iint_{S_1} f(\xi, \tau)[a^2(t-\tau)^2-(x-\xi)^2-z^2]^{-\frac{1}{2}}d\xi d\tau,$$

where S_1 is a segment of a certain hyperbola. The ξ-integration is carried out for the case of a piston, $f(x, t)$ being then independent of x; with the further assumption that the incident wave is uniform beyond a certain distance behind the wave-front, the author sets up an equation of motion for the piston. *F. V. Atkinson* (Oxford).

Kraichnan, Robert H. Electromagnetic analogy to sound propagation in moving media. J. Acoust. Soc. Amer. **27**, 527–530 (1955).

The disturbance of the propagation of sound waves in a fluid medium in motion, due to the fluid motion having a spatial scale long compared with the wave length, has an analog in corresponding phenomena associated with electron optics. To study this analog the author first derives an eikonal equation for sound propagation in a moving medium under the assumption of a low Mach number and small sound amplitude. Further it is assumed that the shear velocity of the fluid motion varies but little over a wave length of the sound. This equation is compared to the equation for a moving electron deduced from the Hamilton-Jacobi equation, simplified by the assumption, that the scalar potential is zero. The coefficients in the two equations may then be identified.

The results may be summed up by saying that, within the limitations noted above, the ray paths of a sound wave in a moving fluid are identical with the trajectories of charged particles in a magnetic field everywhere proportional to the vorticity vector. Finally the author gives some simple illustrations. *H. Bremekamp* (Delft).

Carrier, G. F. The mechanics of the Rijke tube. Quart. Appl. Math. **12**, 383–395 (1955).

L'auteur attire l'attention sur un phénomène très curieux découvert par Rijke [Ann. Phys. Chem. (4) 17, 339–343 (1859)]. Il s'agit d'un tube cylindrique ouvert à ses extrêmités dans lequel on place à une certaine distance du fond un ruban chauffé. Le courant convectif provoque des oscillations acoustiques. Il s'agit d'étudier ces oscillations qui dépendent d'un certain nombre des paramètres. L'auteur étudie en détail l'influence de chacun de ces paramètres. Les résultats obtenus sont en bonne concordance avec les faits observés. *M. Kiveliovitch* (Paris).

Elasticity, Plasticity

★Muschelišvili, N. I. Nekotorye osnovnye zadači matematičeskoĭ teorii uprugosti. Osnovnye uravneniya, ploskaya teoriya uprugosti, kručenie i izgib. [Some fundamental problems of the mathematical theory of elasticity. Fundamental equations, plane theory of elasticity, torsion and bending.] 4th ed. Izdat. Akad. Nauk SSSR, Moscow, 1954. 647 pp. 34.35 rubles.

This edition differs from the 3rd [1949; MR **11**, 626; see MR **15**, 370 for an English translation] only in a number of small changes which do not change the general plan of the book. The text has been reset for this edition.

Cotter, Barbara A., and Rivlin, R. S. Tensors associated with time-dependent stress. Quart. Appl. Math. **13**, 177–182 (1955).

In dieser Arbeit werden die Untersuchungen von Rivlin und Ericksen [J. Rational Mech. Anal. **4**, 323–425 (1955); MR **16**, 881] über allgemeine Materialgleichungen bei endlichen Deformationen fortgeführt; und zwar wird der Fall betrachtet, dass ausser der Spannung t_{ij} noch endlich viele ihrer materiellen zeitlichen Ableitungen $t_{ij}^{(r)}$ auftreten. Es handelt sich also um Gleichungen der Form

$$(1)\qquad f_{pq}(x_{i,\alpha};\ \overset{(1)}{v}_{i,j}, \cdots, \overset{(n)}{v}_{i,j}, t_{ij}, \dot{t}_{ij}, \cdots, \overset{(m)}{t}_{i,j})=0$$

($x_{i,\alpha}$=Verschiebungsgradient, $v_{i}^{(r)}$=Gradient der $(r-1)$-ten materiellen zeitlichen Ableitung der Geschwindigkeit). Aus der Forderung nach Invarianz von (1) gegenüber beliebigen orthogonalen Koordinatentransformationen ergibt sich, dass (1) für isotrope Stoffe auf die Form

$$F_{pq}(C_{ij}, \overset{(1)}{A}_{ij}, \cdots, \overset{(k)}{A}_{ij}, t_{ij}, \overset{(1)}{B}_{ij}, \cdots, \overset{(m)}{B}_{ij})=0, \quad F_{pq}=F_{qp},$$

zurückgeführt werden kann. Dabei ist k die grössere der beiden Zahlen n und m, und die symmetrischen Tensoren $B_{ij}^{(r)}$ sind durch die Rekursionsformel

$$\overset{(0)}{B}_{ij}=t_{ij}, \quad \overset{(r)}{B}_{ij}=\dot{\overset{(r-1)}{B}}_{ij}+\overset{(r-1)}{B}_{lj}\overset{(1)}{v^{l}}_{,i}+\overset{(r-1)}{B}_{il}\overset{(1)}{v^{l}}_{,j}$$

definiert. Die C_{ij} und $A_{ij}^{(r)}$ sind, wie in der oben zitierten Arbeit, durch

$$C_{ij}=x_{i,\alpha}x_{j}{}^{,\alpha}, \quad \overset{(r)}{A}_{ij}=\overset{(r)}{v}_{i,j}+\overset{(r)}{v}_{j,i}+\sum_{s=1}^{r-1}\binom{r}{s}\overset{(r-s)}{v^{l}}_{,i}\overset{(s)}{v}_{l,j}$$

gegeben. *W. Noll* (Los Angeles, Calif.).

Marguerre, K. Ansätze zur Lösung der Grundgleichungen der Elastizitätstheorie. Z. Angew. Math. Mech. **35**, 242–263 (1955). (English, French and Russian summaries)

This primarily expository paper is devoted to the useful task of introducing order in a jungle of isolated, and frequently over-lapping publications: it aims at a presentation of, as well as at an elucidation of the connection between, general solutions of the field equations in the theory of elasticity in terms of stress-functions and displacement-potentials.

The two-dimensional treatment of the plane problem is considered first. Here the two alternative approaches, starting with the Airy stress-function and with the displacement field, respectively, are developed side-by-side. A second section deals, in parallel, with various displacement-potentials appropriate to the general space problem and to the problem characterized by torsionless axisymmetry. The author then turns to his own method of relating the stress-functions of Maxwell and Morera to the potentials of Boussinesq-Papkovich-Neuber. The paper concludes with a section on pure torsion. A brief account of the notation used and a valuable bibliography, containing over fifty items, are appended.

In organizing his material, the author tends to resolve natural conflicts between logical and didactic considerations in favor of the latter. Thus, from the viewpoint of systematic unification, it would seem more satisfying to treat the plane problem, the axisymmetric problem, and the problem of pure torsion, as special cases of the general space problem. Similarly, the discussion of the Maxwell and Morera stress-function which, in its initial stages, applies to any continuous medium, might well have preceded the treatment of topics restricted to elastic media. Furthermore, it seems unfortunate that the important issue of completeness has been handled rather casually throughout the paper.

The author contends that questions of notation are largely a matter of taste. Perhaps so. Some readers, however, are apt to wonder whether the author's mixture of scalar, vector, and dyadic notation, coupled with the avoidance of indicial notation, has resulted in no more than a lack of compactness.

The wide scope of the paper renders certain relevant omissions inevitable. Among these the reviewer notes the absence of a reference to Butty's treatise [Tratado de elasticidad teorica-tecnica, v. I, Centro Estudiantes de Ingenieria de Buenos Aires, 1946; MR 15, 481], which contains comprehensive discussions of the subject matter under consideration. Michell's [Proc. London Math. Soc. **31**, 130–146 (1899), p. 144] integration of the equations governing the rotationally symmetric problem in terms of a single stress-function satisfying a fourth-order partial differential equation, preceded Love's approach to the problem, and should have been mentioned.

At a time when the number of authors probably exceeds the number of readers of papers on elasticity theory, the present survey performs a most welcome, if unrewarding service. It is not the reviewer's purpose to make this service appear less welcome or more unrewarding. *E. Sternberg.*

Tekinalp, Bekir. Large elastic deflections of plane rods. Bull. Tech. Univ. Istanbul **7**, 35–49 (1954). (Turkish summary)

After deriving equations of equilibrium for plane elastic rods, the author introduces a perturbation method for solving problems where the deformed shape and applied force per unit length are given. Some illustrative examples are included. No error estimates or comparisons with exact solutions are given. Truesdell [Proc. 1st Midwestern Conference on Solid Mech., 1953, Univ. of Illinois, 1954, pp. 52–55; MR **15**, 842] has noted that such problems can always be solved by quadratures and has given an analysis of the special case of a plane cantilever loaded normally by a uniform load per unit length. *J. L. Ericksen.*

Lodge, A. S. The transformation to isotropic form of the equilibrium equations for a class of anisotropic elastic solids. Quart. J. Mech. Appl. Math. **8**, 211–225 (1955).

The author determines conditions under which the equilibrium equations of linear elasticity for anisotropic materials can be transformed, by a linear coordinate transformation, into equations formally identical with those for isotropic materials. The 21 elastic moduli must satisfy 14 equations. Transformations effecting this are worked out for one type of symmetry. The results are used to extend Hertz's solution for contact of an elastic solid with a rigid plane to certain anisotropic materials. A general solution for transversely isotropic materials is included, along with the remark that it is not known whether it is complete. Hu [Sci. Sinica **3**, 463–479 (1954); MR **16**, 766] has pointed out that it is not. *J. L. Ericksen* (Washington, D. C.).

Filonenko-Borodič, M. M. On conditions of strength of materials having different tensile and compressive strengths. Inžen. Sb. **19**, 13–36 (1954). (Russian)

The various strength theories establish laws by which it is possible from the response of a body in a tension or compression test to find out the conditions of failure under all possible kind of stresses. Mohr's theory is the only one which takes into account properties of materials having different resistances in tension and compression, and is an exception also in this respect that, when graphical representation of other theories require space coordinates for the three principal stresses σ_x, σ_y, σ_z, Mohr uses plane coordinates for any two principal stresses, or for one principal stress and a shearing stress. Mohr's theory omits always one principal stress. The author's object was to take all principal stresses into account within Mohr's theory. He replaces the two variable stresses in Mohr's equation by two new variables θ_1, θ_2, or θ_2, θ_3, where θ_1, θ_2, θ_3 are symmetric functions of σ_x, σ_y, σ_z. In this way surfaces in $(\sigma_x, \sigma_y, \sigma_z)$-space map curves

in the (θ_1, θ_2)-plane. Assuming specific forms for θ_1, θ_2, θ_3 and various functional relationships between a pair θ_1, θ_2, or θ_2, θ_3, the author obtains different surfaces in $(\sigma_x, \sigma_y, \sigma_z)$-space which could suit different materials. The author uses experimental data of several research workers and shows how they fit his surfaces. This problem is essentially a problem of physically reasonable interpolation between scanty experimental data. The author gives a list of references but omits a very important one, i.e. S. Timoshenko, Strength of materials [v. 2, 2nd ed., Van Nostrand, New York, 1941, pp. 473–482]. *T. Leser.*

Sen, Bibhuti Bhusan. Note on the solution of some problems of semi-infinite elastic solids with transverse isotropy. Indian J. Theoret. Phys. **2**, 87–90 (1954).

Èïdus, D. M. The contact problem of the theory of elasticity. Mat. Sb. N.S. **34**(76), 429–440 (1954). (Russian)
Detailed exposition of results in Dokl. Akad. Nauk SSSR (N.S.) **76**, 181–184 (1954); MR **13**, 465.

Radok, J. R. M. Problems of plane elasticity for reinforced boundaries. J. Appl. Mech. **22**, 249–254 (1955).

Contrary to the method of the indirect approach of certain authors [e.g., Timoshenko, Gurney, Beskin, Wells, Reissner], in this paper, based on Muskhelishvili's monograph on plane elasticity [Some basic problems of the mathematical theory of elasticity, 3rd ed., Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1949; MR **11**, 626; **15**, 370], a general direct method for problems of reinforced cutouts in infinitely thin sheets is deduced. This case is frequent in aircraft construction. The method has the advantage of being applicable to all types of holes which may be, exactly or approximately, mapped conformally onto the unit circle. As an example the problem of the reinforced circular hole is treated in detail numerically. Stress and strain concentration factors are presented and shown graphically. The results of the author's method agree better with experimental results than with those of Reissner and Morduchow [NACA Tech. Note no. **1852** (1949); MR **10**, 649]. Simultaneously an identical method has been used by Savin [Concentration of stresses around openings, Gostehizdat, Moscow-Leningrad, 1951; MR **15**, 370]. *D. P. Rašković* (Belgrade).

Conway, H. D. Further problems in orthotropic plane stress. J. Appl. Mech. **22**, 260–262 (1955).

Using the results of Green [Phil. Mag. (7) **34**, 416–420 (1943); MR **5**, 26] for the problem of an infinite aeolotropic plate containing a circular hole under a uniform tensile stress T with no restriction on the symmetry of the structure, the author first derives results for an orthotropic plate containing a circular hole under a tensile stress T with no restriction on the direction of T with respect to the principal axes of orthotropy of the plate. Then using the results of a previous paper [J. Appl. Mech. **21**, 42–44 (1954); MR **15**, 579] he obtains a solution for the infinite orthotropic plate containing an elliptic hole the axes of which are inclined at an arbitrary angle to the principal axes of orthotropy, and the tension T being in the direction of the minor axis. He then indicates how the solution may be obtained for the most general case, viz., a plate containing an elliptic hole, the plate being subjected to a tension at infinity and the axes of the hole and the direction of the tension both being inclined at arbitrary angles to the principal axes of orthotropy. *R. M. Morris* (Cardiff).

Hieke, Max. Über ein ebenes Distorsionsproblem. Z. Angew. Math. Mech. **35**, 54–62 (1955). (English, French and Russian summaries)

The solution is given of two problems involving the thermal stress field set up in an elastic isotropic disc by a given temperature field. Plane stress is assumed, and the analysis proceeds straightforwardly through the use of stress functions and integrals representing the temperature field. Let the radius of the disc be a and the temperature field be $T(r, \theta)$, where r, θ are plane polar co-ordinates. In both problems T is assumed zero outside a segment of the disc. In the first problem T is constant within the segment, and numerical results are given only for this case. In the second problem T varies within the segment and is given by

$$T(r, \theta) = \begin{cases} 0 & (0 \leqq r \cos\theta < r_1), \\ T_0(r\cos\theta - r_1)/(r_2 - r_1) & (r_1 < r\cos\theta < r_2), \\ T_0 & (r_2 < r\cos\theta \leqq a), \end{cases}$$

where T_0, r_1 and r_2 are constant. *H. G. Hopkins.*

Mitra, Debendranath. On stresses of an isotropic elastic disc in the form of a cardioid rotating steadily in its plane. Z. Angew. Math. Phys. **6**, 136–139 (1955).

Two-dimensional problems of elasticity have been solved by Stevenson [Phil. Mag. (7) **34**, 766–793 (1943); MR **8**, 116], by expressing the stresses and displacements in terms of two complex potential functions. Cases of discs in the form of a circle and of an ellipse rotating steadily have also been solved by him. In this paper the method of Muskhelishvili [Z. Angew. Math. Mech. **13**, 264–282 (1933)] has been adopted and the equations giving the stress functions have been transformed into integral equations by means of the mapping function which represents conformally the region included by the boundary of the disc on a circle. The stress functions are then determined by solving the above integral equations using function-theory. The case for a cardioid is solved as an example. (Author's summary.)
R. M. Morris (Cardiff).

Pestel, E. Eine neue hydrodynamische Analogie zur Torsion prismatischer Stäbe. Ing.-Arch. **23**, 172–178 (1955).

Szmelter, Jan. Solution of the membrane problem by means of the method of finite differences with the use of a special computation device. Rozprawy Inż. **2** (1954), 210–214 (1955). (Polish. Russian and English summaries)

Alfutov, N. A. On a case of a momentary subcritical condition of a cylindrical shell. Prikl. Mat. Meh. **19**, 249–250 (1955). (Russian)

A cylindrical shell is subject to uniform external pressure, and in consequence undergoes a large change in radius. An energy treatment is given of the elastic stability problem on the basis that all transverse cross-sections are similarly distorted. *H. G. Hopkins* (Sevenoaks).

Reissner, Eric. Small rotationally symmetric deformations of shallow helicoidal shells. J. Appl. Mech. **22**, 31–34 (1955).

In the present paper a proposal is made to generalize the known solutions for certain problems of transverse bending and plane stress for circular ring plates, considering a shallow helicoidal shell instead of a flat plate. As might be expected, it is found that the pitch H of the helicoidal middle surface

of the shell is responsible for a coupling of what would be separate problems of plane stress and transverse bending for flat plates. The nature of the theory is seen most clearly by its application to a sample problem for which explicit results are obtained. Let r and φ be plane polar coordinates and let $w_0=H\varphi/2\pi$ be the equation of the middle surface of the helicoidal shell. The author considers a shell with fixed outer edge at $r=b$ and an inner edge $r=a$ which is fixed to a rigid cylinder, displaced an amount δ in the axial direction and rotated an amount ω. To be determined, within the framework of linear, small-deflection theory, are the axial force P and the torque T, per winding of the ring shell, which produce the movements δ and ω. In the limiting case of a flat plate, δ is proportional to P and independent of T, whereas ω is proportional to T and independent of P. For the helicoidal shell we find relations of the form

$$\delta=k_PP+k_{PT}T, \quad \omega=k_TT+k_{TP}P,$$

where the coefficients k contain the influence of the pitch constant H. As in the plate theory explicit solutions in closed form are found for the problems of small deformations which are being considered, but the same is no longer possible for finite deformation, even for flat plates [E. Reissner, Quart. Appl. Math. **10**, 167–173 (1952); MR **14**, 429]. This is in contrast to what occurs for the problems of pure bending and twisting of circular ring plates and shallow helicoidal shells which were considered previously by the same author [ibid. **11**, 473–483 (1954); MR **15**, 482].

R. Gran Olsson (Trondheim).

Reissner, Eric. On transverse vibrations of thin, shallow elastic shells. Quart. Appl. Math. **13**, 169–176 (1955).

According to Marguerrre [Proc. 5th Internat. Congress Appl. Mech., Cambridge, Mass., 1938, Wiley, New York, 1939, pp. 93–101], the vibrations of thin, shallow elastic shells are governed by three simultaneous differential equations in the three displacements. The author has considerably simplified this theory for the case of transverse vibrations by ignoring the longitudinal inertia terms, thus reducing the problem to that of solving two simultaneous differential equations in a stress function and one displacement component. This simplification is justified by an order-of-magnitude analysis, and illustrated by considering the vibrations of a paraboloidal shell with a rectangular boundary.

H. D. Conway (Ithaca, N. Y.).

Vinokurov, S. G. Thermal stresses in plates and shells. Izv. Kazan. Filial. Akad. Nauk SSSR. Ser. Fiz.-Mat. Tehn. Nauk **3**, 18–38 (1953). (Russian)

The preliminary part of this paper gives a general formulation in tensor notation of the equations governing thermal stresses in thin elastic plates and shells. In the remaining part of the paper the solution is given of certain problems associated with a circular plate for which no edge displacement, in either the radial or transverse directions, is permitted. First, an exact analysis is given of the effect on free transverse vibrations due to a uniform change in temperature. Second, Galerkin's method is applied to discuss the behaviour under uniform transverse load when a constant temperature difference is maintained across the plate thickness. In each of these cases the plate is supposed built-in at its edge. Third, an energy method is applied to discuss the behaviour under uniform transverse load with a uniform change in temperature. In this case the plate is supposed simply-supported at its edge.

H. G. Hopkins.

Müller, W., und Krettner, J. Zur Biegungstheorie einer gleichmässig belasteten orthotropen und isotropen Rechteckplatte mit verschiedenen Randbedingungen. Österreich. Ing.-Arch. **9**, 11–21 (1955).

A thin, orthotropic, elastic, rectangular plate with two opposite edges simply-supported is subjected to uniformly-distributed load. The solution of this problem, valid for small transverse displacements, is found for three cases specified by the boundary conditions for the remaining edges. These conditions are (i) one edge simply-supported and the other free, (ii) one edge simply-supported and the other clamped, and (iii) both edges free. Results for isotropic plates are deduced as special cases.

H. G. Hopkins (Sevenoaks).

Favre, Henry, et Schumann, Walter. Etude de la flexion, pour différentes conditions d'appui, des plaques rectangulaires d'épaisseur linéairement variable. Application au cas d'une pression hydrostatique. Bull. Tech. Suisse Romande **81**, 161–173 (1955).

Chien, Wei-Zang, and Yeh, Kai-Yuan. On the large deflection of circular plate. Sci. Sinica **3**, 405–436 (1954).

The paper presents the first two terms of a perturbation series in powers of maximum deflection over plate thickness for a number of problems of finite rotationally symmetric bending of circular plates of uniform thickness. Numerical results are presented in the form of graphs. No references are given to related work, mostly by power series methods, by Federhofer and others.

E. Reissner.

Nowacki, Witold. Statics and dynamics of plates with ribs. Arch. Mech. Stos. **6** (1954), 601–638 (1955). (Polish. Russian and English summaries)

An exact solution of the problems of free and forced vibrations of a rectangular plate with longitudinal and transversal ribs is obtained in this paper.

From the author's summary.

Wierzbicki, Witold. The application of finite differences in two-dimensional states of stress in structures. Rozprawy Inż. **1**, no. 15, 48 pp. (1955). (Polish. Russian and English summaries)

Kane, T. R. Reflection of flexural waves at the edge of a plate. J. Appl. Mech. **21**, 213–220 (1954).

Using Mindlin's equations of flexural motions of plates [J. Appl. Mech. **18**, 31–38 (1951)], the author considers the problem of the reflection of a straight-crested wave at the edge of a semi-infinite plate. These equations accommodate two types of flexural waves and a shear wave. It is found that all of these motions are excited upon incidence of any one of them at a free edge. The character of the reflected waves is influenced by two parameters (the angles of incidence and the ratio of plate thickness to wave length). Incident and emergent waves, various cases of normal and oblique incidences, and grazing incidence are considered too. Similarly to the results of Goodier and Bishop [J. Appl. Phys. **23**, 124–126 (1952); MR **13**, 705], in the latter case the author's equations present an exact solution of the two-dimensional equations of motion.

D. P. Rašković.

Sengupta, A. M. Radial and torsional vibrations of a cylindrically aeolotropic annulus. Indian J. Theoret. Phys. **1**, 125–132 (1953).

Ghosh, S. K. Dynamics of the vibration of a bar exhibiting strain-rate effect under conditions of longitudinal impact. III. Indian J. Theoret. Phys. **1**, 25–40 (1953).

Ghosh, Sudhir Kumar. Energy absorbed by a bar under a compressive impact by an elastic load. IV. Indian J. Theoret. Phys. **1**, 73–78 (1953).

Davidson, J. F. Impact buckling of deep beams in pure bending. Quart. J. Mech. Appl. Math. **8**, 81–87 (1955).

The equations describing lateral instability of a deep beam, first given by L. Prandtl [Kipperscheinungen, Dissertation, Universität München, 1899] and A. G. M. Michell [Phil. Mag. (5) **48**, 298–309 (1899)] are in the present paper extended to the dynamic case in which a beam is suddenly loaded by couples acting in the plane of greatest stiffness. These couples are greater than the first critical moment, and, owing to the slight initial curvature of the beam, their application causes it to deflect exponentially with time until the bending couples are released. For a given value of the bending moment the period of application is determined by the criterion that the amplitude of the subsequent oscillations shall not be excessive. It is found, in most cases, that the resulting impact time is related to the larger of the two gravest component periods of the unloaded beam (torsion and flexure). If the couples are very large, and the gravest component periods widely different, the impact time is fixed by the shorter of the component periods.

R. Gran Olsson (Trondheim).

Seth, B. R. Stability of rectilinear plates. Z. Angew. Math. Mech. **35**, 96–99 (1955). (German, French and Russian summaries)

Die elastische Stabilität der längs Geraden einfach gestützten Platte bei gleichzeitig wirkendem Querdruck und Druck in der Mittelebene der Platte wurde bereits im Falle des Rechtecks sowie des gleichseitigen, rechtwinkligen Dreiecks untersucht [S. Timoshenko, Theory of elastic stability, McGraw-Hill, New York, 1936, S. 327–333] und zwar mit Hilfe trigonometrischer Reihen in zwei Veränderlichen, die für die numerischen Rechnungen nicht immer geeignet sind. Das Problem der elastischen Stabilität der einfach gestützten, geradlinig begrenzten Platte, die nur Randkräften unterworfen wird, kann bekanntlich auf das Problem der Querschwingungen einer gespannten Membran reduziert werden, deren Rand mit dem der Platte übereinstimmt. Der Inhalt dieser Arbeit dient dem Nachweis, dass dies Ergebnis auch gültig bleibt, wenn eine Belastung quer zur Platte wirkt, die als Einzellast, gleichmässig verteilte Last oder als eine Funktion der Koordinaten gegeben sein möge. Bei einer Einzellast wird das Problem exakt auf das einer längs des Randes festgespannten Membran zurückgeführt. Bei einer gleichmässig verteilten Belastung muss der Rand der Membran harmonisch schwingend angenommen werden, weshalb nicht-homogene Lösungen der Schwingungsgleichung erforderlich sind. Die entsprechenden Lösungen für Transversalschwingungen der Membran mit festgehaltenen Rändern sind bereits bekannt [B. R. Seth, Proc. Indian Acad. Sci. Sect. A. **12**, 487–490 (1940); **32**, 421–423 (1950); MR **3**, 123; **14**, 757]. In der Arbeit sind einige kleinere Druckfehler vorhanden.

R. Gran Olsson (Trondheim).

Zav'yalov, V. D., and Timošin, Yu. V. Hodographs of reflected waves for curvilinear boundaries of separation and their interpretation. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1955**, 118–129 (1955). (Russian)

In a layered half-space the boundary between the upper and lower medium is taken in the form of a cylindrical surface (two-dimensional case). Analytical and graphical methods for obtaining a time-distance curve for waves reflected at such an interface are discussed. In these methods based on rules of Geometric Optics it is also shown how to solve the inverse problem, i.e. how to determine the shape of an interface when a time-distance curve is given.

W. S. Jardetzky (New York, N. Y.).

Newlands, Margery. The disturbance due to a line source in a semi-infinite elastic medium with a single surface layer. Philos. Trans. Roy. Soc. London. Ser. A. **245**, 213–308 (1952).

In the first part of this extensive study elastic waves are considered which are generated by a pulse from a source of either longitudinal or shear waves. As usual, the displacement potentials are written in form of integrals, the integrands of which are determined from the boundary conditions. As shown by Bromwich [Proc. London Math. Soc. (2) **15**, 401–448 (1917), pp. 425–430] these integrands can be expanded in negative powers of exponentials. Then, each term of such a series is interpreted as an individual pulse. The evaluation of an integral representing a pulse can be performed by using the appropriate Riemann surface and Sommerfeld contour [Ann. Physik (4) **28**, 665–736 (1909)] in the plane of complex numbers. The potentials and displacements corresponding to forty zero-order and first-order terms have been calculated by the author. Besides the pulses which follow the rules of geometrical optics the discussion shows the existence of the so-called blunt pulses which do not propagate along any minimum-time path. In an approximate evaluation of integrals the method of steepest descent was used.

In the second part a dispersive surface-wave train is considered. Phase- and group-velocity curves are given for Rayleigh waves in the first five modes. A brief section deals with the generation of Love waves. *W. S. Jardetzky.*

Dasgupta, Sushil Chandra. Note on Love waves in a homogeneous crust laid upon heterogeneous medium. II. Indian J. Theoret. Phys. **1**, 121–124 (1953).

This problem can be easily solved for certain particular density and rigidity distributions [see part I, J. Appl. Phys. **23**, 1276–1277 (1952); MR **14**, 517]. In this second note the author considers one more example. If ρ and z vary as $\cosh^2(z/\lambda)$, λ being a constant and z the distance from the interface (at a depth H), the solution is obtained in terms of elementary functions and the period equation takes the form

$$\tan[(c^2/\beta^2-1)^{\frac{1}{2}}\kappa H]=\mu_0[\kappa^2(1-c^2/\beta'^2)+1/\lambda^2]/\mu(c^2/\beta^2-1)^{\frac{1}{2}}$$

(standard notations). *W. S. Jardetzky.*

Sokolovskiĭ, V. V. On the equations of the theory of plasticity. Prikl. Mat. Meh. **19**, 41–54 (1955). (Russian)

Reference is to the problem of the plane strain of a general plastic-rigid material. The material is non-hardening, and therefore obeys a yield condition of the form $F(p, \tau)=0$ where p is the mean compressive stress and τ is the maximum shear stress. This type of material is of importance in theo-

retical studies of soil mechanics. Following the specialisation of the general three-dimensional equations to the present problem, the analysis proceeds in either one of two ways according as the governing equations are elliptic or hyperbolic. Discussion is given of both cases.

Reviewer's note: There is some duplication of previous work [see, e.g., R. Hill, The mathematical theory of plasticity, Oxford, 1950, pp. 294 et seq.; MR **12**, 303].

H. G. Hopkins (Sevenoaks).

Jung, H. Über eine Anwendung der Hillschen Minimalbedingung in der Plastizitätstheorie. Ing.-Arch. **23**, 61–68 (1955).

The expansion of a long cylindrical tube made of an elastic-plastic material has been investigated by so many authors that the subject would seem to be pretty well exhausted. Significant contributions are nevertheless made from time to time. Examples of such contributions are the application of the method of characteristics [R. Hill, E. H. Lee, and S. J. Tupper, Proc. Roy. Soc. London. Ser. A. **188**, 273–289 (1947); MR **8**, 358], the use of Tresca's yield condition and the associated flow rule [W. T. Koiter, Biezeno Anniversary Volume, Stam, Haarlem, 1953, pp. 232–251; MR **14**, 1148], and the present author's use of an extremum principle for work-hardening plastic solids. For an ideal incompressible material, the condition of plane flow requires that the axial stress at a generic point of the expanding tube equal the arithmetic mean of the radial and incircumferential stresses at this point. On account of the elastic compressibility of a real elastic-plastic material, this condition is not strictly fulfilled for such a material. The author therefore writes the axial stress as the sum of the arithmetic mean of the other two principal stresses and a correction term that depends on the radius vector of the considered point and the radius of the elastic-plastic interface. He then uses the minimum principle of Hodge-Prager and Hill [see, e.g., R. Hill, The mathematical theory of plasticity, Oxford, 1950, p. 64; MR **12**, 303] to determine the best expression for a correction term of the assumed form. The general steps leading to the desired result are indicated but the results are not presented in detail. The author reports that the method of characteristics is preferable for treating a single example, but that the present method is superior for the construction of tables of the stresses corresponding to various ratios of the interior and exterior diameters and various ratios of interior pressure to yield stress. *W. Prager.*

Schlechtweg, H. Zur Problematik des Entlastungsvorganges nach plastischer Verformung. Z. Angew. Math. Mech. **35**, 176–183 (1955). (English, French and Russian summaries)

It is generally accepted that the stresses that prevail in an elastic, perfectly plastic solid after full or partial unloading from an elastic-plastic state can be found by superimposing on the stresses of this state the purely elastic response to the reduction in load, provided that the stresses resulting from this superposition do not violate the yield condition. The present author does not seem to feel that this procedure is completely justified. For a thick-walled tube under interior pressure, for example, the analysis described above shows that the slightest reduction in pressure causes the stresses throughout the tube to fall below the yield limit. As a possible alternative to this, the author suggests an unloading process in which an interior annulus remains at the yield limit even though the pressure is being reduced. During this unloading process, the elastic-plastic interface would therefore gradually recede towards the interior surface instead of disappearing as soon as the pressure starts to drop. By a laborious analysis, the author then proceeds to convince himself that this kind of unloading process is not possible. Actually, this investigation is unnecessary, because the procedure indicated in the first sentence of this review certainly gives a state of residual stress that satisfies all equations of the Prandtl-Reuss theory, and the well-known uniqueness theorem of this theory shows that this state is unique. The author briefly mentions this possibility of ruling out alternative unloading processes, but does not follow it up because he has the (unjustified) impression that the proof of the uniqueness theorem presupposes a simply connected domain.

W. Prager (Providence, R. I.).

Grandori Guagenti, Elisa. Campi di esistenza delle varietà carratteristiche nei corpi elasto-plastici. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **18**(**87**), 3–21 (1954).

The paper is based on a stress-strain law of the finite type that was proposed by Locatelli [same Rend. (3) **4**(**73**), 581–598 (1940); MR **8**, 357] and Udeschini [ibid. **5**(**74**), 373–388 (1941); MR **8**, 358] to describe the mechanical behavior of elastic, plastic solids in the domain of contained plastic deformations. The author does not seem to be aware of the serious theoretical objections against all laws of this type [see, e.g., W. Prager, J. Appl. Phys. **19**, 540–543 (1948); **20**, 235–241 (1949); MR **10**, 81, 495]. The application of this law to problems of plane strain, plane stress is discussed. The resulting partial differential equations are of mixed type, the elliptic region corresponding to predominantly elastic and the hyperbolic region to predominantly plastic behavior. Tension, flexure, and torsion are treated as examples. *W. Prager* (Providence, R. I.).

Lomakin, V. A. Elastic-plastic equilibrium of a sphere in an unsteady temperature field. Prikl. Mat. Meh. **19**, 244–248 (1955). (Russian)

A stress-free spherical mass is at a uniform temperature. The surface temperature is suddenly decreased and is kept constant. In this paper attention is directed towards the theoretical analysis of the subsequent elastic-plastic deformation of the mass. Stress equilibrium is assumed, and a finite-strain theory of plasticity is used. The physical constants specifying mechanical behaviour are supposed independent of temperature. Numerical results are presented in graphical form. *H. G. Hopkins* (Sevenoaks).

Levin, E. Indentation pressure of a smooth circular punch. Quart. Appl. Math. **13**, 133–137 (1955).

Reference is to the general problem of the indentation by a punch of a semi-infinite mass made of non-hardening plastic-rigid material which obeys Tresca's yield condition and associated flow rule. The punch is made of rigid material and has a smooth flat base. The normal cross-section of the punch is arbitrary save for restriction that its equation in plane polar co-ordinates r, θ may be written (with suitable choice of origin) in the form $r=f(\theta)$ where f is a single-valued continuous function of θ. This includes the case of a punch whose cross-section is concave towards interior points. A kinematically admissible velocity field is constructed for the general case, and this field is specialised to the particular case of a circular punch. An application of

the Drucker-Prager-Greenberg upper-bound limit-analysis theorem then gives $5.84k$ (where k is the yield stress in shear) as an upper bound for the average indentation pressure p. Previous work of Shield and Drucker [J. Appl. Mech. **20**, 453–460 (1953)] gives $5k$ as a lower bound to p.

Reviewer's note: An exact solution of the circular-punch problem is due to R. T. Shield (in press), and the exact value of p is $5.69k$. Hence Levin's result is only $+2\frac{1}{2}\%$ in error. The error is believed to be least for the case of the circular punch and to be much greater for other cases, e.g. that of a rectangular punch. *H. G. Hopkins.*

Wang, A. J., and Prager, W. Plastic twisting of a circular ring sector. J. Mech. Phys. Solids **3**, 169–175 (1955).

The fully plastic twisting of a circular ring sector is a problem of importance for the determination of the plastic behavior of closely coiled helical springs. The ring sector discussed in the paper is of an arbitrary solid cross section and it is composed of an ideally plastic material. For a circular cross section this problem has been treated by Freiberger [Commonwealth of Australia, Aero. Res. Lab. Rep. **SM. 213** (1953)]. A geometrical interpretation of Freiberger's equation leads to a simple construction of the characteristics. The shear lines, which are the orthogonal trajectories of the characteristics, can thus be constructed for a cross section of arbitrary shape. The authors' analysis differs from Freiberger's with respect to the circumferential velocity, which produces the warping of the cross sections. Freiberger assumes that this velocity vanishes along the line of stress discontinuity. It is shown in the paper that the shear rate must vanish along the line of stress discontinuity. This condition and the equation satisfied by the warping function enable the authors to determine the rate of warping of the cross section. To illustrate the developed methods square and rectangular sections are treated as examples.

E. T. Onat (Ankara).

Grubin, A. N., and Lihačev, Yu. I. Analysis of the stressed state arising in the stage of large plastic deformations in stretching cylindrical specimens with ring-shaped groove. Ž. Tehn. Fiz. **25**, 512–528 (1955). (Russian)

Analysis is given of the problem of the progressive quasi-static extension under axial force of a circular cylinder with a transverse groove cut along its surface. The problem is therefore one of rotational symmetry, and all quantities are functions of two space co-ordinates and a time co-ordinate specifying the progress of the deformation. The method is based upon a finite-strain theory of plasticity. At the outset a power series development in an axial co-ordinate is assumed for the components of displacement, the coefficients in these series being unknown functions of a radial co-ordinate. The method proceeds through various approximations. In particular, approximations occur in connection with the above series and with the satisfaction of the boundary conditions at the progressively-deforming groove. No application is made to any specific problem.

Reviewer's note. This problem is of considerable technological importance. Unfortunately an analysis based upon an incremental theory of plasticity would be prohibitively difficult at the present time. The circumstances in which results predicted by the present analysis will be sufficiently reliable for design use are to be determined only through direct comparison with experimental data.

H. G. Hopkins (Sevenoaks).

Mikeladze, M. Š. Elastic-plastic deformations in rapidly rotating discs of variable thickness. Inžen. Sb. **15**, 21–34 (1953). (Russian)

This paper considers the problem of the title in relation to the design of turbine rotors. The analysis, necessarily approximate, is based upon the concept of plane stress and use is made of tangent moduli. Illustrative numerical examples are worked. Reviewer's note. More elaborate studies of this problem based upon the finite-strain theory of plasticity have been made by M. H. Lee Wu [NACA Rep. no. **1021** (1951); MR **13**, 406] and M. Zaid [J. Aero. Sci. **20**, 369–377 (1953); MR **16**, 311]. *H. G. Hopkins.*

Berio, Angelo. I fenomeni elasto-plasto-viscosi nei solidi. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **18(87)**, 295–355 (1954).

The paper begins with a survey of stress-strain laws proposed by various authors to describe the isothermal mechanical behavior of solids. The present author seems to object to the fact that each of these laws stresses some effects and neglects others. He proposes instead a single law that incorporates elastic after-effect, viscosity, and plasticity including work-hardening and Bauschinger effect.

W. Prager (Providence, R. I.).

Lee, E. H. Stress analysis in visco-elastic bodies. Quart. Appl. Math. **13**, 183–190 (1955).

This paper concerns isotropic linear visco-elastic bodies with a constitutive equation of the form

$$Ps_{ik}=Qe_{ik}, \quad P'\sigma=Q'\epsilon, \tag{1}$$

where s_{ik} is the stress deviator, e_{ik} the strain deviator, σ and ϵ are the traces of the stress and strain tensor, respectively, and P, Q, P', Q' are polynomials in the differential operator with respect to time. Deformations are assumed to be infinitesimal, and inertia forces due to deformation are neglected.

The author shows that problems in visco-elasticity can be reduced formally to problems in linear elasticity by applying the Laplace transform to (1) as well as to the boundary conditions and equilibrium equations. This method turns out to be particularly appropriate to problems where the time dependence of the boundary tractions and body forces separates out with a common factor ("proportional loading"). An alternative method is developed which also involves Laplace transform and which is convenient in the case of moving surface tractions. As illustrative examples the case of a point force variable in time and acting normally on the surface of a semi-infinite body and the case of such a force moving along a straight line on the surface are considered. *W. Noll* (Los Angeles, Calif.).

Head, A. K., and Louat, N. The distribution of dislocations in linear arrays. Austral. J. Phys. **8**, 1–7 (1955).

J. D. Eshelby, F. C. Frank und F. R. N. Nabarro [Phil. Mag. (7) **42**, 351–364 (1951); MR **12**, 882] haben das Problem von einer Reihe von geraden und untereinander gleich grossen Versetzungen, welche in der selben Gleitebene enthalten sind und sich infolge ihrer eigenen Abstossung und einer äusseren Schubspannung verteilen, theoretisch untersucht. Die Verfasser der vorliegenden Arbeit geben zur Behandlung solcher Probleme eine annähernde jedoch viel einfachere Methode an, deren Grundgedanke ist, dass sie die diskreten Versetzungen durch eine kontinuierliche Verteilung von verschmierten infinitesimalen Versetzungen mit dem selben Burgersschen Vektor ersetzen. Das dabei auf-

tretende mathematische Problem ist die Lösung der Integralgleichung

$$(1)\qquad \int_D \frac{f(x)}{x-x_0}dx - \frac{T(x_0)}{A} = \frac{P(x_0)}{A},$$

wo $f(x)$, die Verteilung der Versetzungen, die sich entlang der X-Achse bewegen können, die gesuchte Funktion ist. $P(x)$ bedeutet die angewandte Spannung, $T(x)$ ist eine Spannung von sehr kurzer Reichweite (Diracsche δ-Funktion), die es verhindert, dass die Versetzungen sich ins Unendliche entfernen und A ist ein aus den elastischen Konstanten gebildeter Ausdruck, der für Stufen- und Schraubenversetzungen verschiedene Werte hat. In (1) ist aus physikalischen Gründen der Cauchysche Hauptwert des Integrals zu nehmen; für $T(x)=0$ erhält man dann die Lösung von (1) nach einer von N. I. Muskhelishvili [Singuläre Integralgleichungen, Gostehizdat, Moskau-Leningrad, 1946, S. 259; MR 11, 523; 15, 434] angegebenen Methode. Als Beispiele werden mehrere physikalisch interessante Probleme besprochen. *T. Neugebauer* (Budapest).

MATHEMATICAL PHYSICS

Landolt, Max. Réponse de l'auteur de *Grandeur, Mesure et Unité* à la critique présentée par M. M. Eskenazi. Bull. Tech. Univ. Istanbul 7, 79–80 (1954).

Eskenazi, M. Réplique de M. M. Eskenazi. Bull. Tech. Univ. Istanbul 7, 81–82 (1954).

See same Bull. 5 (1952), 17–26 (1953); MR 16, 426.

Kratzer, A. Physik und Mathematik. Studium Gen. 6, 619–628 (1953).

Magnus, A. Mathematik und ihre Anwendung in der Chemie. Studium Gen. 6, 629–637 (1953).

Optics, Electromagnetic Theory

Have

***Sommerfeld, Arnold. Optics. Lectures on theoretical physics, Vol. IV.** Translated by O. Laporte and P. A. Moldauer, Academic Press Inc., New York, 1954. xiii+383 pp. $6.80.

This book has its origin in lectures on Optics delivered by Sommerfeld to university students around 1934. It is divided into the following six chapters: I) Reflection and refraction of light. II) Optics of moving media and sources. Astronomical topics. III) Theory of dispersion. IV) Crystal optics. V) The theory of diffraction. VI) Addenda, chiefly to the theory of diffraction. There is also a historical introduction, noting the main landmarks in the development of Optics. Some problems and their solutions are also included.

The treatment presents Optics from the outset as a branch of Electromagnetic Theory. Quantum concepts are introduced in Chapter II, after a brief excursion into the Theory of Relativity. (A much fuller treatment of Relativity is given in the preceding volume of this series.) Throughout, the stress is on physical principles rather than on specialized techniques. Topics such as the geometrical optics of instruments (e.g. Gaussian optics and geometrical theory of aberrations), as opposed to the eikonal equation and Fermat's principle, are hardly touched upon. It is evident, however, that some omissions must arise in a treatment which covers the limited space available. Within this space, the selection of topics could hardly be improved upon, if a sound exposition of theoretical principles is the main objective.

In discussing diffraction near the focal point in §45, Sommerfeld bases his treatment on a paper by Debye [Ann. Physik (4) 30, 755–776 (1909)] and stresses the rigorous nature of the solution. The discussion would appear to support the widely held view (erroneous in the reviewer's opinion) that Debye's solution leads to a more accurate description of the field in the region of focus than a treatment based on the Huyghens-Kirchhoff theory. An analysis of this problem, using Huyghens' principle, was given by E. Lommel [ibid. (2) 25, 643–655 (1885)] and by H. Struve [Mém. Acad. Imp. Sci. St.-Pétersbourg (7) 34, no. 5 (1886)]. It would be out of place here to discuss the relative merits of these solutions, but it seems appropriate to say that the difference lies not so much in the degree of rigour as in the formulation: Lommel and Struve solved the physical problem approximately, whilst Debye solved an idealized problem exactly; and in regions where the Debye integrals have been evaluated (along the axis and in the focal plane), the Debye and the Lommel-Struve solutions give identical results.

But these are only small points in a masterly exposition of Optics, characterized by clarity and lucidity of presentation. This is a book which is ideally suited for the use of university students and teachers alike. *E. Wolf.*

Straškevič, A. M., and Gluzman, N. G. Aberrations of relativistic electron beams. Ž. Tehn. Fiz. 24, 2271–2284 (1954). (Russian)

An electron beam is defined as the totality of moving electrons having the same initial velocity and trajectories close to an axial trajectory. Relativistic equations of motion are derived for a wide beam in an arbitrary electrostatic field. In this treatment the axis need not be a straight line but may be bent, e.g. in the form of a helix. The systems with a bent axis are of special interest since they may give rise to negative chromatic aberration. The special cases considered are the relativistic equations of motion in an axially symmetrical field, a flat field, the field of a cylindrical lens, and the field of a cylindrical condenser. Image aberrations are computed for the relativistic case for an axially symmetrical lens and a cylindrical lens. *J. E. Rosenthal.*

Straškevič, A. M. Extension of the electron-optical theory of deflecting electrostatic systems to the case of relativistic particles. Ž. Tehn. Fiz. 24, 2264–2270 (1954). (Russian)

The electrostatic systems considered have two planes of symmetry perpendicular to each other. The electrostatic field is symmetrical with respect to one plane and antisymmetrical with respect to the other, i.e. the electrodes producing the field are charged to the potentials $U+u$ and $U-u$ respectively. The trajectories of fast paraxial electrons through such a system are determined from relativistic equations of motion. The results are carried out to the second order of approximation. The deflection sensitivity of the system is determined to the order of u/U and $(u/U)^2$ and found to be higher than in the non-relativistic case. Two cases are given as illustrations: the deflecting system of a parallel plate condenser and the system of two electrodes in

the same plane. End effects are disregarded for entering as well as for emerging electrons. The results are intended for application, e.g. to cathode-ray tube design, but their usefulness is impaired by the failure to consider means for considering end effects on entering electrons, such as the Deserno effect [Arch. Elektrotech. 29, 139–148 (1935)].
J. E. Rosenthal (Passaic, N. J.).

Teisseyre, R. The diffraction of a dipole field by a perfectly conducting wedge. Bull. Acad. Polon. Sci. Cl. III. 3, 157–162 (1955).

The diffraction problem of an arbitrary electric or magnetic dipole by a perfectly conducting wedge is solved by generalizing a method due to Senior [Quart. J. Mech. Appl. Math. 6, 101–114 (1953); MR 14, 933]. Details are promised in a later paper. *A. E. Heins* (Pittsburgh, Pa.).

Mikaèlyan, A. L. Electromagnetic waves in a rectangular wave guide filled with a magnetized ferrite. Doklady Akad. Nauk SSSR (N.S.) 98, 941–944 (1954). (Russian)

Normal modes are first determined for a slab $0\leqq z\leqq z_0$ bounded by ideally conducting planes and filled by anisotropic ferrite magnetized uniformly parallel to the z-axis. Equations are then set up determining combinations of four such modes which can be propagated in the region $0\leqq x\leqq x_0$, $0\leqq z\leqq z_0$, all four walls being assumed ideally conducting.
F. V. Atkinson (Oxford).

★Mikaelyan, A. L. Electromagnetic waves in a rectangular waveguide filled with a magnetized ferrite. Morris D. Friedman, Russian Translation, Two Pine Street, West Concord, Mass., 1954. 7 pp. (mimeographed). $3.50.

Translation of the paper reviewed above.

Vilenskiĭ, I. M. On the influence of the magnetic field of the earth upon the interaction of radio-waves in the ionosphere. Ž. Eksper. Teoret. Fiz. 26, 42–56 (1954). (Russian)

The author extends his previous work [same Ž. 22, 544–561 (1952); MR 15, 487] upon the Luxemburg effect to take into account the influence of the earth's magnetic field.
B. Friedman (Berkeley, Calif.).

Clemmow, P. C., and Heading, J. Coupled forms of the differential equations governing radio propagation in the ionosphere. Proc. Cambridge Philos. Soc. 50, 319–333 (1954).

For fields whose time variation is exp (ipt), Maxwell's equations are

$$\operatorname{curl}\mathbf{E}=-ip\mu_0\mathbf{H},\quad \operatorname{curl}\mathbf{H}=ip\mathbf{D}.$$

In the presence of the earth's magnetic field and a vertically (z-direction) stratified ionosphere, $\mathbf{D}=\epsilon_0\mathbf{E}+\epsilon_0 M\mathbf{E}$, where M is the susceptibility tensor of the ionosphere. Assume that the field components are independent of y and depend on x solely through the factor exp $(-ikx\sin\theta)$ where k is the free-space propagation constant; then E_z and H_z may be eliminated from Maxwell's equations to obtain

$$(1)\qquad \frac{d\mathbf{e}}{dz}=-ikT\mathbf{e},$$

where $\mathbf{e}$ is a column vector whose components are E_x, $-E_y$, Z_0H_x, Z_0H_y (Z_0 is the characteristic impedance of free space), and where T is a matrix depending on M and θ.

The authors introduce the transformation $\mathbf{f}=R^{-1}\mathbf{e}$ into (1), where R is a matrix such that $R^{-1}TR$ is a diagonal matrix Δ, thus obtaining

$$(2)\qquad \frac{d\mathbf{f}}{dz}+ik\Delta\mathbf{f}=-R^{-1}R'\mathbf{f}.$$

The equations (2) are said to be coupled because the right-hand side contains no derivatives of $\mathbf{f}$ and because, for a homogeneous medium, the right-hand side reduces to zero and the resulting equations each contain only one component of $\mathbf{f}$. Equations (2) are then solved by successive approximations. [Similar results have been given by H. B. Keller and J. B. Keller, New York Univ., Washington Square Coll. Arts Sci., Math. Res. Group, Res. Rep. No. **EM-33** (1951); and H. B. Keller, ibid. Nos. **EM-56** (1953); **EM-57** (1953); MR 13, 346; 15, 585.]

The authors show also that in the case of a horizontal magnetic field or in case the plane of incidence is perpendicular to the plane of the magnetic meridian the equations (2) can be re-written as a pair of second-order coupled equations. Finally, the authors consider the case in which a pair of upgoing and downgoing waves is propagated independently of another such pair. Mathematically, this corresponds to the fourth-order system separating into two independent second-order systems. *B. Friedman.*

Budden, K. G. A reciprocity theorem on the propagation of radio waves via the ionosphere. Proc. Cambridge Philos. Soc. 50, 604–613 (1954).

In the space below the ionosphere, let an upgoing plane wave with its wave normal at an angle θ to the vertical be resolved into plane-polarized components, one with the electric field in the plane of propagation and the other with the electric field perpendicular to the plane of propagation. Let a and b, respectively, be the coefficients of these components when suitably normalized. Similarly, for a downcoming wave let the coefficients of the component in the plane and the component perpendicular to the plane be c and d, respectively. Consider the reflection matrix $R(\theta)$ such that

$$\begin{pmatrix}c\\d\end{pmatrix}=R(\theta)\begin{pmatrix}a\\b\end{pmatrix}.$$

The author uses the paper reviewed above to prove the following reciprocity theorem: When the plane of propagation contains the magnetic meridian, then $R(-\theta)=$transpose $R(\theta)$. *B. Friedman* (Berkeley, Calif.).

Azbel', M. Ya. Conductivity of films in a longitudinal magnetic field. Dokl. Akad. Nauk SSSR (N.S.) 99, 519–522 (1954). (Russian)

Expressions are derived for the electronic distribution function in a metallic film in a constant magnetic field parallel to the surface of the film. For the sake of simplicity, the electric and magnetic fields are assumed to be parallel so that there is no Hall effect. However, the same method may be used to derive the distribution function in crossed electric and magnetic fields. While the latter result is not given, the author states that it indicates an error in the derivation of the distribution function by MacDonald and Sarginson in the case where a constant $\alpha<2$, α depending on the electron energy, the thickness of the film, the magnetic field strength, and the effective mass [Proc. Roy. Soc. London. Ser. A. 203, 223–240 (1950)]. Formulas are derived for the effective conductivity valid for all values of q, the

coefficient of reflection of electrons from the surface. Since the value $q=0$ is significant physically, it is considered in detail. The rather involved expression for the conductivity is evaluated for a sufficiently strong as well as a sufficiently weak magnetic field, both for thin and for thick films. The dependence of the conductivity on the magnetic film is plotted and analyzed for different film thicknesses.

J. E. Rosenthal (Passaic, N. J.).

Azbel', M. Ya. On the theory of skin-effect in a constant magnetic field. Dokl. Akad. Nauk SSSR (N.S.) **100**, 437–440 (1955). (Russian)

The surface impedance is determined in a constant magnetic field parallel to the surface of the metal on the assumption of diffuse reflection of electrons from the surface. The treatment presupposes complete familiarity with the paper reviewed above. Since the general expression for the impedance is quite involved, particular attention is given to the two limiting cases of low and high frequencies respectively where the evaluation may be carried out. The impedance at low frequencies agrees with that found for the classical skin-effect. The evaluation of the impedance at high frequencies is restricted, for the sake of simplicity, to strong magnetic fields. The results are compared with those obtained for a constant magnetic field perpendicular to the surface of the metal. [M. Ya Azbel' and M. I. Kaganov, same Dokl. (N.S.) **95**, 41–44 (1954)]. The application of the formulas to experimental results is discussed very briefly.

J. E. Rosenthal (Passaic, N. J.).

Roberts, P. H. On the reflection and refraction of hydromagnetic waves. Astrophys. J. **121**, 720–730 (1955).

Das Problem, dass eine ebene harmonische hydromagnetische Welle auf die Trennungsebene von zwei Medien, die beide eine unendliche Leitfähigkeit besitzen, einfällt, wird theoretisch weiter untersucht. Den Ausgangspunkt der Berechnungen bilden die bekannten Differentialgleichungen der Magneto-Hydrodynamik, die eine Folge der Maxwellschen Differentialgleichungen und der Eulerschen Gleichungen der Hydrodynamik sind. Die Schwierigkeit, dass diese Gleichungen nichtlinear sind, wird mit Hilfe der bekannten Annahme umgangen, dass ein konstantes Magnetfeld $\mathbf{H}_0$ existiert, über dem ein kleines Störfeld $\mathbf{h}$ überlagert ist. Weiter wird ein Hilfsvektor $\mathbf{b}$ mit Hilfe der Gleichungen $\mathbf{h}=\mathbf{H}_0\partial\mathbf{b}/\partial\zeta$, $\mathbf{u}=\partial\mathbf{b}/\partial t$ und $\operatorname{div}\mathbf{b}=0$ eingeführt, wo ζ die Koordinate entlang der Richtung von $\mathbf{H}_0$ und $\mathbf{u}$ die Geschwindigkeit der Flüssigkeitsteilchen bedeutet. Für $\mathbf{b}$ folgt dann die Differentialgleichung

$$(1) \qquad \frac{\partial^2 b}{\partial t^2}=V^2\frac{\partial^2 b}{\partial \zeta^2}-\operatorname{grad}\tilde{\omega},$$

wo $V^2=H_0^2/4\pi\rho$ ist und $\tilde{\omega}$ einen aus dem Druck, aus der Dichte (ρ) und dem magnetischen Felde gebildeten Ausdruck bedeutet. (1) wird mit Hilfe des Ansatzes $\mathbf{b}=\mathbf{a}+\operatorname{grad}\psi$ integriert, in dem ψ einer speziellen Differentialgleichung und der solenoidale Vektor $\mathbf{a}$ einer einfachen Wellengleichung genügen. Aus den an der Trennungsfläche auftretenden Grenzbedingungen und teilweise mit Hilfe von in der Optik üblichen Gedankengängen folgen dann endlich die Gleichungen

$$(2) \qquad \cot\theta_1=\cot\theta_0+2\tan\beta\cos\gamma,$$

und

$$(3) \qquad \cot\theta_2=\frac{V_1}{V_2}\cot\theta_0+\left(\frac{V_1}{V_2}-1\right)\tan\beta\cos\gamma,$$

wo θ_0, θ_1 und θ_2 den Einfalls-, den Reflexions- und den Brechungswinkel bedeuten, V_1 und V_2 die Ausbreitungsgeschwindigkeiten in den zwei Medien sind und die Winkel β und γ die Polarkoordinaten der Richtung von $\mathbf{H}_0$ bedeuten. (2) und (3) wurden schon von V. C. A. Ferraro [Astrophys. J. **119**, 393–406 (1954); MR **15**, 761] für den speziellen Fall, dass die Richtung der Polarisation parallel zur Trennungsebene liegt, hergeleitet. Hier wird deren Gültigkeit ganz allgemein gezeigt. Die erhaltenen Resultate werden auch tabellarisch zusammengestellt und zuletzt wird das Problem der Reflexion an einer freien Oberfläche behandelt.

T. Neugebauer (Budapest).

Lampariello, Giovanni. Una soluzione rigorosa delle equazioni di Minkowski dell'elettrodinamica dei corpi in moto e sua interpretazione fisica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17**, 100–108 (1954).

The author investigates plane electromagnetic waves in a moving medium (isotropic and homogeneous), using Minkowski's equations. [Since these equations are Lorentz-invariant, it appears to the reviewer that the results may be obtained more simply by applying a Lorentz transformation to waves in the medium at rest. See review of earlier paper, same Rend. (8) **17**, 37–44 (1954): MR **16**, 775. In that review, however, the criticism regarding characteristics was based on a misreading and must be withdrawn.]

J. L. Synge (Dublin).

Arakawa, T. Analysis of an approximation of high degree by conformal mapping of strip-shaped, elliptic, and rectangular two-wire feeder lines. Rep. Univ. Electro-Commun. **5**, 171–192 (1953). (Japanese. English summary)

In analyzing characteristics on two-wire feeder lines in cases enumerated in the title, the author replaces mapping functions by approximately equivalent ones of elementary nature. Numerical comparison is made. *Y. Komatu.*

Arakawa, T. A method for correcting edge effect for certain vis-a-vis conductors. Rep. Univ. Electro-Commun. **5**, 147–170 (1953). (Japanese. English summary)

An approximate method for correcting edge effect for vis-a-vis strip-shaped and rectangular conductors. It is based upon replacing respective conformal maps of Schwarz-Christoffel type by those which can be expressed elementarily in explicit forms. *Y. Komatu* (Tokyo).

Gonzalez del Valle, A. The synthesis of multipoles, chief problem of cybernetics, for the metric geometry of nets and previous theorems. Calc. Automat. y Cibernet. **4**, no. 10, 1–8 (1955). (Spanish. English summary)

★**Bayard, M. Théorie des réseaux de Kirchhoff. Régime sinusoidal et synthèse.** Editions de La Revue d'Optique, Paris, 1954. xv+408+5 pp. 3200 francs.

Among books on the general subject which have appeared to date, this treatise contains the most comprehensive treatment of the analysis and synthesis of linear passive networks. It contains many new results not available in the comparable work of W. Cauer [Theorie der linearen Wechselstromschaltungen, Akademische Verlagsgesellschaft, Leipzig, 1941], especially in n-terminal pair synthesis.

Chapters 1–4 contain introductory material on network analysis such as complex representation of currents, reso-

nance, Kirchhoff's laws etc. Ch. 5 gives an exposition of matrix algebra and Ch. 6 discusses networks from a graph-theoretic viewpoint. Other network concepts such as superposition principle, reciprocity law, generalized impedance and admittance and duality are given in Chapters 7 and 8. These chapters cover 103 pages.

Beginning with Ch. 9, the author develops systematically techniques for and results on network synthesis. The author studies in Ch. 9 "open networks" (réseaux ouverts), which include all $2n$-terminal networks, and considers in Ch. 10 transformations on réseaux ouverts such that with respect to these transformations certain quantities such as driving-point impedance between terminal pairs remain invariant. Ch. 10 also includes a discussion of energy relations in linear networks. Ch. 11 discusses transformers and their role in network synthesis. Ch. 12 introduces the notion of scattering matrices for n terminal pairs, leading to an extension of Richards' theorem [Duke Math. J. **14**, 777–786 (1947); MR **9**, 181] to positive real matrices. Some special networks are considered in Chapters 13 and 14 and characteristics of general 4-poles and of transmission lines are given in Chapters 15 and 16. These chapters (9–16) cover 127 pages.

Ch. 17 initiates analytic network-synthesis theory proper. It begins with a study of properties of positive real functions and positive real matrices and includes such results as Foster's reactance theorem [Bell System Tech. J. **3**, 259–267 (1924)], conditions for stability and the author's and Bode's integral formulas [H. W. Bode, Network analysis and feedback amplifier design, Van Nostrand, New York, 1945] involving real and imaginary parts of positive real functions. Chapters 18 and 19 contain a discussion of pure reactance dipole synthesis due to Foster and Cauer and Cauer's extension to resistance-inductance and resistance-capacitance networks [Arch. Elektrotech. **17**, 355–388 (1926)]. Ch. 20 discusses general dipole synthesis and includes such results as Brune's [J. Math. Phys. **10**, 191–235 (1931)] and Darlington's [ibid. **18**, 257–353 (1939); MR **1**, 275] synthesis methods and the method developed by Bott and Duffin [J. Appl. Phys. **20**, 816 (1949); MR **12**, 307]. Synthesis of networks whose elements are themselves dipoles and synthesis of artificial lines are discussed in Chapters 21 and 22 respectively. Ch. 23 is on network synthesis of reduced matrices (matrices réduites) and includes results by Gewertz [J. Math. Phys. **12**, 1–257 (1933)] and the author. The last two chapters (24 and 25) discuss more recent developments on $2n$-pole synthesis and synthesis with a minimum number of elements. These two chapters contain recent contributions by Belevitch, Leroy, Oono, Yasuura, Tellegen, MacMillan and the author. Chapters 17 to 25 cover 158 pages.

The book is concisely written and includes many exercises.

C. Y. Lee (Chatham, N. J.).

Duffin, R. J. Impossible behavior of nonlinear networks. J. Appl. Phys. **26**, 603–605 (1955).

This note is devoted to the proof of the following principle: an isothermal electro-mechanical system whose primary resistors are quasi-linear cannot convert direct current to alternating current. Here a resistor is said to be quasi-linear if the voltage-drop across it is a function of the current alone, and if the differential resistance is positive. The proof leans on the earlier investigations of the author in this subject.

R. Bott (Princeton, N. J.).

✶Cherry, E. Colin. Generalized concepts of networks. Proceedings of the symposium on information networks, New York, April, 1954, pp. 175–184. Polytechnic Institute of Brooklyn, Brooklyn, N. Y., 1955.

This introductory paper discusses the uses to which networks have been put in various disciplines; from flow charts, topological graphs and trees, to sociograms. *R. Bott.*

Twiss, R. Q. Nyquist's and Thevenin's theorems generalized for nonreciprocal linear networks. J. Appl. Phys. **26**, 599–602 (1955).

This is a brief discussion of the equivalence of a general linear reciprocal network with internal noise sources to a source-free network with certain correlated nodal noise current generators by means of a generalization of Nyquist's theorem. Conditions for realizability of passive nonreciprocal networks are also desired. *E. Weber.*

✶Schröder, Hans. Vierpoltheorie und erweiterte Zweipoltheorie. Fachbuchverlag GmbH, Leipzig, 1954. 191 pp. DM 9.50.

The first third of this book gives a relatively simple treatment of steady-state transmission-line theory from the engineering point of view. Towards the end of the section, the conventional hyperbolic functions are introduced, and particularly applied to loaded lines which naturally form the transition to symmetrical fourpoles or two-terminal-pair networks. Many numerical examples serve to illustrate various conditions of power transfer for a.c. and d.c. sources, but all are restricted to symmetrical fourpoles or the combination of sources as active twopoles with passive twopoles.

The last section treats linear two-pole networks (one-terminal-pair networks) with stress upon simple duality principles. The book is intended as an introduction for students of the new five-year program for engineers. For some reason there is complete absence of non-German references and only the name Thévenin occurs in connection with the theorem deduced from Helmholtz's writings. Though Foster's reactance theorem is demonstrated, his name does not appear. *E. Weber* (Brooklyn, N. Y.).

Hott, François. Synthèse des quadripôles passifs. Rev. Gén. Elec. **39**, 203–205 (1955).

It is demonstrated that any realisable passive fourpole can be represented by a single Tee-section with a shunt admittance in cascade, or by a single π-section with a series impedance in cascade. In general, this means that one can compute for any given frequency four equivalent twopole parameters. The general computation is carried through by using the matrix of the general fourpole parameters which is particularly adapted to cascade connections.

E. Weber (Brooklyn, N. Y.).

LaRosa, R., and Carlin, Herbert J. A general theory of wideband matching with dissipative 4 poles. J. Math. Phys. **33**, 331–345 (1955).

Given a generator with pure resistive internal impedance and an arbitrary load, the authors examine the interposition of lossy four-pole networks which give a desired power transfer over a wide frequency band. By means of the scattering matrix the general conditions for optimum match are derived, i.e., for the maximum scale factor with which the power transfer function can be achieved. Comparison with an optimum lossless matching network indicates that the optimum lossy network can never be inferior in power

transfer efficiency by more than 3 db. An instructive example illustrates the application of the design principles.
E. Weber (Brooklyn, N. Y.).

Burks, Arthur W., McNaughton, Robert, Pollmar, Carl H., Warren, Don W., and Wright, Jesse B. Complete decoding nets: general theory and minimality. J. Soc. Indust. Appl. Math. 2, 201–243 (1954).

The important results contained in this paper are concerned with conjunction $b-d$ decoding nets. They are defined as nets of logical conjunction elements having no feedback whose inputs are bracketed into d sets of b inputs each. In an allowed net state only one input in each set takes the logical value 1 and an output exists which assumes the value 1 in only that state. Cost of a net is defined as equal to the number of inputs to logical elements appearing in the net.

Three common designs of such nets are described and analysed as to cost. The last of these, called a $b-d$ balanced m.s.n., is shown to have minimum cost among conjunction $b-d$ decoding nets. An exhaustive enumeration of all minimal nets of this type is achieved by also including certain variations on the balanced m.s.n. design.
D. E. Muller.

Quantum Mechanics

Have

***Mandl, F. Quantum mechanics.** Academic Press Inc., New York; Butterworths Scientific Publications, London, 1954. viii+233 pp. $5.80.

This book is based on a series of lectures given by the author to a group of experimentalists at Harwell and is intended to bring out the unifying mathematical scheme underlying quantum mechanics.

The first five chapters deal with the principles and mathematical formalism of quantum mechanics: some mathematical concepts and methods, the Schroedinger equation and the physical concepts connected with it, some examples of energy eigenfunctions and eigenvalues, the general principles of quantum mechanics, and the fundamentals of matrix mechanics. The remaining four chapters discuss some applications of the general theory to more specific problems: many-particle systems and their angular momentum and symmetry properties, time-independent perturbations, collision processes, and finally an introduction to group theory and some of its applications. A large number of exercises, as well as hints for solving them, are included.

The book is rather concise. It appears suitable for someone who has already had at least a little contact with quantum mechanics. No great rigor is sought, and for the more elaborate proofs reference is made to other sources. For the experimental physicist who would like to get an over-all picture of the conceptual and mathematical structure of quantum mechanics this book should be very good. For the theoretician it may serve well as a stepping-stone to the larger and more rigorous treatises. *N. Rosen* (Haifa).

Sugawara, Masao. The mass variation with velocity in Bopp's unitary field theory. I. Progr. Theoret. Phys. 7, 303–316 (1952).

F. Bopp [Z. Naturf. 1, 53–58, 196–203, 237–242 (1946); MR 7, 539; 8, 425, 426] proposed a non-local modification of the electromagnetic field equations, involving a Lorentz invariant "action-at-a-distance function" $\epsilon(x)$. On the basis of the energy-momentum density tensor obtained by Bopp and by Heisenberg [ibid. 5a, 251–259 (1950); MR 12, 573], calculations are carried out, in the case of a point charge moving with constant velocity, for the total energy and momentum of the field. These have the proper dependence on velocity if $\epsilon(x)$ satisfies a certain condition.
N. Rosen (Haifa).

Sugawara, Masao, and Minami, Sakae. The mass variation with velocity in the Bopp's unitary field theory. II. Progr. Theoret. Phys. 7, 563–572 (1952).

Carrying out the same kind of calculation as in the paper reviewed above, but using the expression for the energy-momentum density tensor found by Y. Ôno [Progr. Theoret. Phys. 6, 925–938 (1951); MR 14, 117], the authors obtain the same results as before. They also verify that the non-local field of A. Pais and G. E. Uhlenbeck [Phys. Rev. (2) 79, 145–165 (1950); MR 12, 227] give satisfactory results for the energy and momentum of a point charge with constant velocity. *N. Rosen* (Haifa).

Kar, K. C., Sanatani, S., and Bhattacharyya, R. K. A simple derivation of Klein-Nishina formula without matrices. Indian J. Theoret. Phys. 2, 49–68 (1954).

Minami, Sakae. On a non-local interaction in the quantum electrodynamics. J. Fac. Sci. Hokkaido Univ. Ser. II. 4, 153–165 (1952).

Let

$$\bar{A}_\mu(x)=\int dx' F(x-x')A_\mu(x'),$$

$$\bar{\psi}(x)=\int dx' G(x-x')\psi(x'),$$

where F and G are "delta-like" invariant functions [see McManus, Proc. Roy. Soc. London. Ser. A. 195, 323–336 (1948); MR 10, 664]. Let the field equations be

$$(\gamma_\mu\partial/\partial x_\mu+m)\psi(x)=ie\bar{A}_\mu(x)\gamma_\mu\bar{\psi}(x),$$
$$\Box A_\mu(x)=-ie\bar{\bar{\psi}}(x)\gamma_\mu\psi(x).$$

Assuming power-series expansions like $\psi=\sum e^n\psi^{(n)}$ for the opeators ψ, $\bar{\psi}$ etc. [Källén, Ark. Fys. 2, 187–194 (1950); MR 12, 570], the field equations can be solved to any desired order in e^n. The author calculates, in particular, the expressions for vacuum polarization and electron self-energy in the second order and shows that these can be made finite by an appropriate choice of the form factors F and G.
A. Salam (Cambridge, England).

Minami, Sakae, and Sugawara, Masao. On renormalization in the field theory with non-localized interaction. J. Fac. Sci. Hokkaido Univ. Ser. II. 4, 166–172 (1952).

It is shown that the conventional renormalization procedure applies to the solutions of the field equations proposed in the paper reviewed above up to the second order in charge. *A. Salam* (Cambridge, England).

Epstein, Saul T. Derivation of the Feynman-Dyson rules from time independent theory. Phys. Rev. (2) 98, 196–198 (1955).

A simple derivation of Feynman-Dyson rules [R. P. Feynman, Phys. Rev. (2) 76, 769–789 (1949); MR 11, 765; F. J. Dyson, ibid. (2) 75, 486–502 (1949); MR 10, 418] for writing matrix elements is given, starting with the conventional time-independent quantum-mechanical scattering theory. *A. Salam* (Cambridge, England).